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Gaudin-type models, non-skew-symmetric classical r-matrices and nested Bethe ansatz

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Abstract

We consider quantum integrable systems associated with the Lie algebra $gl(n)$ and Cartan-invariant non-dynamical non-skew-symmetric classical r-matrices. We describe the sub-class of Cartan-invariant non-skew-symmetric r-matrices for which exists the standard procedure of the nested Bethe ansatz associated with the chain of embeddings $gl(n) \supset gl(n-1) \supset gl(n-2) \supset \cdots \supset gl(1)$. We diagonalize the corresponding quantum integrable systems by its means. We illustrate the obtained results by the examples of the generalized Gaudin systems with and without external magnetic field associated with three classes of non-dynamical non-skew-symmetric classical r-matrices.

1. Introduction

Quantum integrable spin models with long-range interaction play important role in the non-perturbative physics. They are applied in the theory of small metallic grains [1–4], where the famous Richardson's model [5] is used, nuclear physics [6,7], theory of colored Fermi gases [8], in the so-called "central spin model" theory [9,10], etc. The main examples of such the integrable models are the famous Gaudin spin models [11] associated with simple (reductive) Lie

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algebras $\mathfrak{g}$ and skew-symmetric $\mathfrak{g} \otimes \mathfrak{g}$-valued classical r-matrices with spectral parameter and their integrable modifications — Gaudin models in external magnetic fields [12].

In our previous papers [13,14] we have proposed a generalization of classical and quantum Gaudin models with [14] and without [13] external magnetic field associated with arbitrary non-skew-symmetric $\mathfrak{g} \otimes \mathfrak{g}$-valued non-dynamical classical r-matrices with spectral parameters that satisfy the so-called generalized or "permuted" classical Yang–Baxter equation. Moreover, we have shown that the corresponding models are applied in order to construct new integrable fermion models of reduced BCS-type [15] and integrable proton–neutron Richardson's-type models of nuclear physics [16]. This makes a study of the generalized Gaudin models associated with non-skew-symmetric classical r-matrices important not only from the mathematical but also from the physical point of view.

In the present paper we consider the problem of the diagonalization of the generating functions of commutative integrals of quantum integrable models associated with non-skew-symmetric classical r-matrices. We concentrate on the case $\mathfrak{g} = gl(n)$, $n > 2$ (the case $n = 2$ was in detail considered in our previous paper [15]) and "diagonal" in the root basis classical r-matrices. We develop a general theory, which is applicable for all types of quantum integrable models whose Lax algebra possesses a representation with a vacuum vector, but in the examples we concentrate on the generalized Gaudin models with and without magnetic field. The examples of other types of quantum integrable systems associated with the non-skew-symmetric r-matrices will be considered elsewhere.

The problem of a simultaneous diagonalization of the integrable Hamiltonians is a complicated problem in the case of the Lie algebras of a higher rank. The standard approach for the diagonalization of the usual Cartan-symmetric Gaudin models in such the cases is the "nested" or "hierarchical" Bethe ansatz invented in [17] for the quantum group case and repeated in [18] for the Lie-algebraic case in the subcase of skew-symmetric Cartan-invariant classical r-matrices. The main idea of the nested or hierarchical algebraic Bethe ansatz is to reduce the problem of the diagonalization of the generating functions of the commutative integrals of $gl(n)$ to the same problem for the Lie algebra $gl(n-1)$ and then apply this method recursively, using the hierarchy of embedded subalgebras $gl(n) \supset gl(n-1) \cdots \supset gl(1)$. In the present paper we find a class of the non-skew-symmetric classical r-matrices for which such a method is applicable and construct the corresponding Bethe vectors, Bethe equations and answers for the spectrum of the generating function of the quantum integrals of the first and second order in the quantum dynamical variables. Quite remarkably that, like in the case of the Lie algebra $gl(2)$ [15], all the answers are written in general form without the specification of the dependence of the r-matrix on the spectral parameters (modulo the fact that the corresponding r-matrix satisfies generalized classical Yang–Baxter equation and two additional constraints), and without the specification of the model (i.e. without the specification of the eigen-value of the Lax operator on the vacuum vector).

We illustrate the obtained results on the concrete examples of the generalized Gaudin models with and without external magnetic field associated with (non-skew-symmetric) shifted rational, shifted trigonometric and certain shifted Z_2-graded classical r-matrices [19]. The first two classes of the r-matrices constitute n-parametric deformations of standard rational and trigonometric r-matrices. The third class of the r-matrix is one-parametric deformation of the "twisted" or "Z_2-graded" rational r-matrices [20]. Let us emphasize, that the r-matrices considered in the present paper does not satisfy usual classical Yang–Baxter equation and lie out of the Belavin–Drinfeld classification [21].

We would like also to outline that *general* non-skew-symmetric r-matrices are not connected with the quantum groups or related structures. The only class of the classical non-skew-symmetric r-matrices associated with such the structures are those related to A–B–C–D algebras of [22] (see also [23]). That is why it is methodologically important to develop a theory of the corresponding quantum integrable systems independently of (much better elaborated) quantum-group formalism [24,25].

In the end of the Introduction let us mention also the recent papers [26,27], where Gaudin-type models based on the Lie algebras $gl(3)$ and $gl(4)$ and their Bethe ansatz solutions were considered. Contrary to our paper, these articles deal only with the cases of skew-symmetric rational [27] and skew-symmetric rational and trigonometric r-matrices [26]. The Bethe ansatz results for the Gaudin models based on such the r-matrices were at first obtained in [18].

The structure of the present paper is the following: in the second section we remind general facts about quantum integrable systems associated with non-skew-symmetric classical r-matrices, in the third section we consider nested Bethe ansatz (for such the r-matrices) and prove the main theorem. At last in the fourth section we present three classes of examples of the classical r-matrices for which our construction is applicable.

2. Quantum integrable systems and r-matrices

2.1. Definitions and notations

Let $\mathfrak{g} = gl(n)$ be the Lie algebra of the general linear group over the field of complex numbers. Let X_{ij}, $i, j = \overline{1, n}$ be a standard basis in $gl(n)$ with the commutation relations:

$$[X_{ij}, X_{kl}] = \delta_{kj} X_{il} - \delta_{il} X_{kj}. \tag{1}$$

Definition 1. A function of two complex variables $r(u_1, u_2) = \sum_{i,j,k,l=1}^{n} r_{ij,kl}(u_1, u_2) X_{ij} \otimes X_{kl}$ with values in the tensor square of the algebra $\mathfrak{g} = gl(n)$ is called a classical r-matrix if it satisfies the following generalized classical Yang–Baxter equation [28–30]:

$$\big[r^{12}(u_1, u_2), r^{13}(u_1, u_3)\big] = \big[r^{23}(u_2, u_3), r^{12}(u_1, u_2)\big] - \big[r^{32}(u_3, u_2), r^{13}(u_1, u_3)\big], \tag{2}$$

where $r^{12}(u_1, u_2) \equiv \sum_{i,j,k,l=1}^{n} r_{ij,kl}(u_1, u_2) X_{ij} \otimes X_{kl} \otimes 1$, $r^{13}(u_1, u_3) \equiv \sum_{i,j,k,l=1}^{n} r_{ij,kl}(u_1, u_3) \times X_{ij} \otimes 1 \otimes X_{kl}$, etc., and $r_{ij,kl}(u, v)$ are matrix elements of the r-matrix $r(u, v)$.

It is easy to show that there are three classes of equivalences in the space of solutions of Eq. (2). They are:

1. “gauge transformations”: $r(u_1, u_2) \to Ad_{g(u_1)} \otimes Ad_{g(u_2)} r(u_1, u_2)$.
2. “reparametrization”: $r(u_1, u_2) \to r(v_1, v_2)$, where $u_i = u_i(v_i)$, $i \in 1, 2$.
3. “rescaling”: $r(u_1, u_2) \to f(u_2) r(u_1, u_2)$, where function $f(u_2)$ is arbitrary.

Remark 1. In the case of skew-symmetric r-matrices, i.e. when $r^{12}(u_1, u_2) = -r^{21}(u_2, u_1)$, where $r^{21}(u_2, u_1) = P^{12} r^{12}(u_1, u_2) P^{12}$ and P^{12} interchanges the first and second spaces in tensor product, the generalized classical Yang–Baxter equation reduces to the usual classical Yang–Baxter equation [21,31,32]:

$$\big[r^{12}(u_1, u_2), r^{13}(u_1, u_3)\big] = \big[r^{23}(u_2, u_3), r^{12}(u_1, u_2) + r^{13}(u_1, u_3)\big]. \tag{3}$$

Observe, that gauge transformations and reparametrizations are equivalences in the spaces of solutions also of Eq. (3), while "rescaling" is not, because it does not preserve skew-symmetry property of the r-matrix.

In the present paper we are interested only in the r-matrices that by the equivalence transformations may be brought to the form possessing the following decomposition:

$$r(u_1, u_2) = \frac{\Omega}{u_1 - u_2} + r^0(u_1, u_2), \tag{4}$$

where $\Omega = \sum_{i,j=1}^{n} X_{ij} \otimes X_{ji}$ is the tensor Casimir and $r^0(u_1, u_2)$ is a regular on the "diagonal" $u = v$ function with values in $gl(n) \otimes gl(n)$.

For the subsequent we will also need the following three definitions:

Definition 2. We will call classical r-matrix to be $\mathfrak{g}_0 \subset gl(n)$-invariant if

$$\left[r(u_1, u_2), X \otimes 1 + 1 \otimes X\right] = 0, \quad \forall X \in \mathfrak{g}_0. \tag{5}$$

Observe, that on the level of Lie groups this definition means exactly the G_0-invariance:

$$(Ad_g \otimes Ad_g) \cdot r(u_1, u_2) = r(u_1, u_2),$$

where $g \in G_0$ and G_0 is a Lie group of the algebra $\mathfrak{g}_0$.

Definition 3. We will call classical $gl(n)$-valued r-matrix to be diagonal in the root basis if it has the following form:

$$r(u_1, u_2) \equiv \sum_{i,j=1}^{n} r_{ji}(u_1, u_2) X_{ij} \otimes X_{ji}. \tag{6}$$

Diagonal in the root basis classical r-matrices are automatically Cartan-invariant. Inverse is not true: not all Cartan-invariant r-matrices are diagonal in the root basis.

Remark 2. Observe that in the case of the diagonal r-matrices the generalized classical Yang–Baxter equation (2), is rewritten in the component form as follows:

$$r_{ij}(u_1, u_2) r_{jl}(u_1, u_3) - r_{il}(u_1, u_2) r_{jl}(u_2, u_3) - r_{il}(u_1, u_3) r_{ij}(u_3, u_2) = 0, \tag{7}$$

where $i, j, l \in \overline{1, n}$ and all three indices do not coincide.

Definition 4. A $gl(n)$-valued function $c(u) = \sum_{i,j=1}^{n} c_{ij}(u) X_{ij}$ of one complex variable is called "generalized shift element" if it satisfies the following equation:

$$\left[r^{12}(u_1, u_2), c(u_1) \otimes 1\right] - \left[r^{21}(u_2, u_1), 1 \otimes c(u_2)\right] = 0. \tag{8}$$

In the subsequent we will be interested only in the diagonal shift elements, i.e. shift elements of the following form:

$$c(u) = \sum_{i=1}^{n} c_{ii}(u) X_{ii}.$$

Remark 3. Observe, that for skew-symmetric classical r-matrices any element of its symmetry algebra is a shift element. For non-skew-symmetric r-matrices it is not true.

Remark 4. Observe, that the "shift element" takes its values in $gl(n)$ and is pertinent to the Lax operators (see Proposition 2.1 below). It should not be confused with a "shift" of the r-matrix itself, which is certain constant element of $gl(n) \otimes gl(n)$ (see Ref. [19] and examples in Section 4).

2.2. Algebra of Lax operators and classical r-matrices

Using the classical r-matrix $r(u_1, u_2)$ it is possible to define the following "tensor" Lie bracket:

$$\big[\hat{L}(u_1) \otimes 1, 1 \otimes \hat{L}(u_2)\big] = \big[r^{12}(u_1, u_2), \hat{L}(u_1) \otimes 1\big] - \big[r^{21}(u_2, u_1), 1 \otimes \hat{L}(u_2)\big], \tag{9}$$

where $\hat{L}(u) = \sum_{i,j=1}^{n} \hat{L}_{ij}(u) X_{ij}$, $r^{21}(u_2, u_1) = P^{12} r^{12}(u_1, u_2) P^{12}$.

The tensor bracket (9) between the quantum Lax matrices $\hat{L}(u_1)$ and $\hat{L}(u_2)$ is a symbolical expression of the Lie brackets between their matrix elements. In the case of the diagonal r-matrices (6), which are the main object of interest in the present article, the corresponding commutation relations (9) have the following simple form:

$$\begin{aligned}\big[\hat{L}_{ij}(u_1), \hat{L}_{kl}(u_2)\big] = {} & \delta_{il}\big(r_{kl}(u_1, u_2)\hat{L}_{kj}(u_1) + r_{ij}(u_2, u_1)\hat{L}_{kj}(u_2)\big) \\ & - \delta_{kj}\big(r_{kl}(u_1, u_2)\hat{L}_{il}(u_1) + r_{ij}(u_2, u_1)\hat{L}_{il}(u_2)\big).\end{aligned} \tag{10}$$

The explicit form of the operators $\hat{L}_{ij}(u)$ as functions of the auxiliary spectral parameter u is not arbitrary. It agrees with the structure of the r-matrix $r(u_1, u_2)$ and depends on the concrete quantum system under consideration. In the present paper we will consider the simplest but the most important examples of such the systems.

Let us now clarify the role of the shift elements in the algebra of Lax operators. The following proposition holds true:

Proposition 2.1. *Let $\hat{L}(u)$ be the Lax operator satisfying the commutation relations (9) and $c(u)$ be the shift element satisfying Eq. (8). Then the operator*

$$\hat{L}^c(u) \equiv \hat{L}(u) + c(u)$$

also satisfies the commutation relations (9).

(Proof of this proposition follows from the explicit form of commutation relations (9), definition (8) and the fact that $c^{ij}(u)$ are c-numbers.)

2.3. Quantum integrals

In this subsection we will explain the connection of classical non-skew-symmetric $gl(n) \otimes gl(n)$-valued r-matrices with quantum integrability. It will be shown that, just like in the case of classical r-matrix Lie–Poisson brackets [29], the Lie bracket (9) leads to an algebra of mutually commuting quantum integrals.

Let us consider the following linear and quadratic functions in generators of the Lax algebra:

$$\hat{t}_n(u) = \mathrm{tr}_n\, \hat{L}(u) = \sum_{i=1}^{n} \hat{L}_{ii}(u) \quad \text{and} \quad \hat{\tau}_n(u) = \frac{1}{2}\,\mathrm{tr}_n\, \hat{L}^2(u) = \frac{1}{2}\sum_{i,j=1}^{n} \hat{L}_{ij}(u)\hat{L}_{ji}(u). \tag{11}$$

The following important theorem holds true:

Theorem 2.1. *Let $\hat{L}(u)$ be the Lax operator satisfying the commutation relations (9) with the diagonal in the root basis classical r-matrix. Assume that in some open region $U \times U \subset \mathbb{C}^2$ the function $r(u,v)$ possesses the decomposition (4). Then the operator-valued functions $\hat{\tau}_n(u)$ and $\hat{t}_n(u)$ are a generators of a commutative algebra, i.e.:*

$$\left[\hat{t}_n(u), \hat{t}_n(v)\right] = 0, \qquad \left[\hat{\tau}_n(u), \hat{t}_n(v)\right] = 0, \qquad \left[\hat{\tau}_n(u), \hat{\tau}_n(v)\right] = 0.$$

(The proof of the theorem involves the Leibnitz rule and Jacobi identity for the commutator, the $gl(n)$-invariance of the corresponding quadratic form and some consequences of the generalized classical Yang–Baxter equations. It essentially repeats the same proof for the $gl(2)$ case [15].)

Remark 5. It is necessary to emphasize that, generally speaking, operator

$$\hat{t}_n(u) = \mathrm{tr}_n\, \hat{L}(u) = \sum_{i=1}^{n} \hat{L}_{ii}(u)$$

does not belong to a center of the algebra of Lax operators. Indeed, even in the simplest case of the diagonal r-matrices from the commutation relations (10) we obtain:

$$\left[\hat{t}_n(u), \hat{L}_{kl}(v)\right] = \big(r_{ll}(v,u) - r_{kk}(v,u)\big)\hat{L}_{kl}(v). \tag{12}$$

This expression is not zero if $r_{kk}(u,v) \neq r_{ll}(u,v)$.

From Theorem 2.1 follows the next corollary:

Corollary 2.1. *Let the points ν_l belong to the open region U. Then all the operators of the form $\hat{H}_{\nu_l} = -\operatorname{res}_{u=\nu_l} \hat{\tau}_n(u)$ and $\hat{C}_{\nu_m} = -\operatorname{res}_{u=\nu_m} \hat{t}_n(u)$, $l, m \in \overline{1, N}$ mutually commute.*

Remark 6. Observe that the operators $\hat{H}_{\nu_l}$ are exact analogs of the generalized Gaudin Hamiltonians [13] and coincide with them for the special choice of the Lax operator. From the equality (12) it is easy to deduce that the operators $\hat{C}_{\nu_l}$ belong to the center of the algebra of Lax operators.

To summarize: we have constructed an algebra of quantum commutative operators that coincide with a quantization of linear and quadratic subalgebra of the algebra of Lie–Poisson commuting integrals of some classical integrable system admitting Lax representation. The problem of quantization of the other “higher” integrals is very complicated. It is solved only in the partial case of the classical rational r-matrices (i.e. r-matrices for which $r^0(u,v) \equiv 0$) in [33]. Fortunately, for the physically applications the most important are linear and quadratic integrals. Moreover, as we will show below, their diagonalization does not depend on the higher integrals. That is why we do not consider the problem of the quantization of the higher integrals here.

2.4. Example: generalized Gaudin systems

Let us now consider the most important for the applications (and the simplest at the same time) examples of quantum integrable systems associated with classical r-matrices.

Let $\hat{S}_{ij}^{(m)}$, $i, j = \overline{1,n}$, $m = \overline{1,N}$ be linear operators in some Hilbert space that span Lie algebra isomorphic to $gl(n)^{\oplus N}$ with the commutation relations:

$$\big[\hat{S}_{ij}^{(m)}, \hat{S}_{kl}^{(p)}\big] = \delta^{pm}\big(\delta_{kj}\hat{S}_{il}^{(m)} - \delta_{il}\hat{S}_{kj}^{(m)}\big). \tag{13}$$

Let us fix N distinct points of the complex plain ν_m, $m = 1, 2, \ldots, N$. It is possible to introduce the following quantum Lax operator [13]:

$$\hat{L}(u) = \sum_{i,j=1}^{n} \hat{L}_{ij}(u)X_{ij} \equiv \sum_{m=1}^{N}\sum_{i,j=1}^{n} r_{ij}(\nu_m, u)\hat{S}_{ji}^{(m)}X_{ij}. \tag{14}$$

Using generalized classical Yang–Baxter equation it is possible to show that it satisfies a linear r-matrix algebra (9). This quantum Lax operator is a Lax operator of the generalized $gl(n)$-valued Gaudin spin chains.

For the applications important are also the shifted Lax operators [14]:

$$\hat{L}^{c}(u) = \sum_{i,j=1}^{n} \hat{L}_{ij}(u)X_{ij} \equiv \sum_{m=1}^{N}\sum_{i,j=1}^{n} r_{ij}(\nu_m, u)\hat{S}_{ji}^{(m)}X_{ij} + \sum_{i,j=1}^{n} c_{ij}(u)X_{ij}. \tag{15}$$

This quantum Lax operator is a Lax operator of the generalized $gl(n)$-valued Gaudin spin chains in an external magnetic field and a generalized shift element $c(u) = \sum_{i,j=1}^{n} c_{ij}(u)X_{ij}$ plays a role of an external nonhomogeneous magnetic field.

As it was remarked above, residues of generating function $\hat{t}_n(u)$ produces only trivial integrals proportional to the Casimir functions $\hat{C}_{\nu_k} = \sum_{i=1}^{n} S_{ii}^{(k)}$. Residues of the second order generating function $\hat{\tau}(u)$ produces nontrivial integrals $\hat{H}_{\nu_l}$ [13]:

$$\hat{H}_{\nu_l} = \sum_{k=1, k\neq l}^{N}\sum_{i,j=1}^{n} r_{ij}(\nu_k, \nu_l)\hat{S}_{ji}^{(k)}\hat{S}_{ij}^{(l)} + \frac{1}{2}\sum_{i,j=1}^{n}\big(r_{ji}^{0}(\nu_l, \nu_l) + r_{ij}^{0}(\nu_l, \nu_l)\big)\hat{S}_{ij}^{(l)}\hat{S}_{ji}^{(l)} \tag{16}$$

in the case of the “unshifted” Lax operators and

$$\begin{aligned}\hat{H}_{\nu_l}^{c} = {} & \sum_{k=1, k\neq l}^{N}\sum_{i,j=1}^{n} r_{ij}(\nu_k, \nu_l)\hat{S}_{ji}^{(k)}\hat{S}_{ij}^{(l)} \\ & + \frac{1}{2}\sum_{i,j=1}^{n}\big(r_{ji}^{0}(\nu_l, \nu_l) + r_{ij}^{0}(\nu_l, \nu_l)\big)\hat{S}_{ij}^{(l)}\hat{S}_{ji}^{(l)} + \sum_{i=1}^{n} c_{ii}(\nu_l)\hat{S}_{ii}^{(l)}\end{aligned} \tag{17}$$

in the case of “shifted” Lax operators [14]. Here $r_{ij}^{0}(u, v)$ denotes the regular part of the component function $r_{ij}(u, v)$ of the r-matrix $r(u, v)$.

The Hamiltonians (16) are the generalized Gaudin Hamiltonians corresponding to the diagonal $gl(n) \otimes gl(n)$-valued r-matrix and Hamiltonians (17) are the generalized Gaudin Hamiltonians in magnetic field corresponding to the same diagonal $gl(n) \otimes gl(n)$-valued r-matrix and diagonal external magnetic field.

3. Nested Bethe ansatz

The main idea of the nested or hierarchical algebraic Bethe ansatz proposed for the quantum-group case in [17], and in the Lie-algebraic case (in the subcase of skew-symmetric r-matrices) in [18], is to reduce the problem of the diagonalization of the generating functions of the commutative integrals of $gl(n)$ to the same problem for the Lie algebra $gl(n-1)$ and then apply this method recursively, using the hierarchy of embedded subalgebras $gl(n) \supset gl(n-1) \supset \cdots \supset gl(1)$. In this section we will show for what class of the non-skew-symmetric classical r-matrices such the method is applicable and construct the corresponding Bethe vectors, Bethe equations and answers for the spectrum of the generating function of the quantum integrals of the first and second order.

3.1. Diagonalization

Let us diagonalize $\hat{\tau}_n(u)$ in a certain representation space V with the help of the nested algebraic Bethe ansatz. Let V be a space of an irreducible representation of the algebra of Lax operators. Let us assume that there exist a vacuum vector $\boldsymbol{\Omega} \in V$ such that:

$$\hat{L}_{ii}(u)\boldsymbol{\Omega} = \Lambda_{ii}(u)\boldsymbol{\Omega}, \qquad \hat{L}_{kl}(u)\boldsymbol{\Omega} = 0, \quad i,k,l \in \overline{1,n},\ k > l \tag{18}$$

and the whole space V is generated by the action of $\hat{L}_{kl}(u)$, $k < l$ on the vector $\boldsymbol{\Omega}$.

The following theorem holds true:

Theorem 3.1. *Let the r-matrix $r(u,v)$ satisfy the condition (4) and the following two conditions:*

$$\text{(i)} \quad r_{ji}(u,v) = r_{-,i}(u,v), \qquad r_{ij}(u,v) = r_{+,i}(u,v)$$
$$\forall j \in \overline{i+1,n},\ \forall i \in \overline{1,n-1}, \tag{19a}$$

$$\text{(ii)} \quad r_{22}(u,v) = r_{33}(u,v) = \cdots = r_{nn}(u,v) \tag{19b}$$

or

$$\text{(ii)}' \quad \partial_v r_{22}(u,v) = \partial_v r_{33}(u,v) = \cdots = \partial_v r_{nn}(u,v)$$
$$\text{and} \quad \left(r^0_{-,i}(u,u) + r^0_{+,i}(u,u)\right) = 0, \quad \forall i \in \overline{1,n-1}. \tag{19c}$$

Then the spectrum of the generating function $\hat{\tau}_n(u)$ in the space V has the form:

$$\begin{aligned}
2\Lambda_n(u) = {} & \sum_{k=1}^{n}\left(\Lambda_{kk}(u) + \sum_{j=1}^{M_{k-1}} r_{kk}\big(v_j^{(k-1)},u\big) - \sum_{j=1}^{M_k} r_{kk}\big(v_j^{(k)},u\big)\right)^2 \\
& + \sum_{k=1}^{n}\left(\partial_u + \left(r^0_{-,k}(u,u) + r^0_{+,k}(u,u)\right)\right) \\
& \times \left((n-k)\Lambda_{kk}(u) - \sum_{i=k+1}^{n}\Lambda_{ii}(u) + (n-k)\sum_{j=1}^{M_{k-1}} r_{kk}\big(v_j^{(k-1)},u\big)\right) \\
& - \sum_{k=1}^{n}(n-k+1)\sum_{j=1}^{M_k} r_{-,k}\big(v_j^{(k)},u\big) r_{+,k}\big(v_j^{(k)},u\big),
\end{aligned} \tag{20}$$

where $M_0 = M_n = 0$, M_k, $k \in \overline{1,n-1}$ are non-negative integers, $r_{-,n}(u,v) \equiv r_{+,n}(u,v) \equiv 0$ and “rapidities” $v_i^{(k)}$, $k \in \overline{1,n-1}$, $i \in \overline{1,M_k}$ satisfy the following Bethe-type equations:

$$
\begin{aligned}
&\Lambda_{11}\big(v_i^{(1)}\big) - \Lambda_{22}\big(v_i^{(1)}\big) - \big(r_{11}^0\big(v_i^{(1)}, v_i^{(1)}\big) + r_{22}^0\big(v_i^{(1)}, v_i^{(1)}\big)\big) \\
&\quad + \frac{n}{2}\big(r_{-,1}^0\big(v_i^{(1)}, v_i^{(1)}\big) + r_{+,1}\big(v_i^{(1)}, v_i^{(1)}\big)\big) \\
&\quad - \frac{n-2}{2}\big(r_{-,2}^0\big(v_i^{(1)}, v_i^{(1)}\big) + r_{+,2}\big(v_i^{(1)}, v_i^{(1)}\big)\big) \\
&= \sum_{j=1;\, j\neq i}^{M_1} \big(r_{11}\big(v_j^{(1)}, v_i^{(1)}\big) + r_{22}\big(v_j^{(1)}, v_i^{(1)}\big)\big) - \sum_{j=1}^{M_2} r_{22}\big(v_j^{(2)}, v_i^{(1)}\big),
\end{aligned}
\tag{21a}
$$

$$
\begin{aligned}
&\Lambda_{k+1k+1}\big(v_i^{(k+1)}\big) - \Lambda_{k+2k+2}\big(v_i^{(k+1)}\big) - \big(r_{k+1k+1}^0\big(v_i^{(k+1)}, v_i^{(k+1)}\big) \\
&\quad + r_{k+2k+2}^0\big(v_i^{(k+1)}, v_i^{(k+1)}\big)\big) \\
&\quad + \frac{n-k}{2}\big(r_{-,k+1}^0\big(v_i^{(k+1)}, v_i^{(k+1)}\big) + r_{+,k+1}\big(v_i^{(k+1)}, v_i^{(k+1)}\big)\big) \\
&\quad - \frac{n-k-2}{2}\big(r_{-,k+2}^0\big(v_i^{(k+1)}, v_i^{(k+1)}\big) + r_{+,k+2}^0\big(v_i^{(k+1)}, v_i^{(k+1)}\big)\big) \\
&= \sum_{j=1;\, j\neq i}^{M_{k+1}} \big(r_{k+1k+1}\big(v_j^{(k+1)}, v_i^{(k+1)}\big) + r_{k+2k+2}\big(v_j^{(k+1)}, v_i^{(k+1)}\big)\big) \\
&\quad - \sum_{j=1}^{M_k} r_{k+1k+1}\big(v_j^{(k)}, v_i^{(k+1)}\big) - \sum_{j=1}^{M_{k+2}} r_{k+2k+2}\big(v_j^{(k+2)}, v_i^{(k+1)}\big), \quad k \in \overline{1, n-3},
\end{aligned}
\tag{21b}
$$

$$
\begin{aligned}
&\Lambda_{n-1n-1}\big(v_i^{(n-1)}\big) - \Lambda_{nn}\big(v_i^{(n-1)}\big) - \big(r_{n-1n-1}^0\big(v_i^{(n-1)}, v_i^{(n-1)}\big) + r_{nn}^0\big(v_i^{(n-1)}, v_i^{(n-1)}\big)\big) \\
&\quad + r_{-,n-1}^0\big(v_i^{(n-1)}, v_i^{(n-1)}\big) + r_{+,n-1}\big(v_i^{(n-1)}, v_i^{(n-1)}\big) \\
&= \sum_{j=1;\, j\neq i}^{M_{n-1}} \big(r_{n-1n-1}\big(v_j^{(n-1)}, v_i^{(n-1)}\big) + r_{nn}\big(v_j^{(n-1)}, v_i^{(n-1)}\big)\big) \\
&\quad - \sum_{j=1}^{M_{n-2}} r_{n-1n-1}\big(v_j^{(n-2)}, v_i^{(n-1)}\big),
\end{aligned}
\tag{21c}
$$

where $r_{\pm,i}^0(u,v)$, $r_{ii}^0(u,v)$, $i \in \overline{1,n}$ denote regular parts of the functions $r_{\pm,i}(u,v)$ and $r_{ii}(u,v)$ correspondingly.

Remark 7. Observe, that in the formulas above for the spectrum and Bethe equations it is necessary to take into account the condition (ii) or (ii)$'$, i.e. to put either $r_{22}(u,v) = r_{33}(u,v) = \cdots = r_{nn}(u,v)$ or $\partial_v r_{22}(u,v) = \partial_v r_{33}(u,v) = \cdots = \partial_v r_{nn}(u,v)$ and $(r_{-,i}^0(u,u) + r_{+,i}^0(u,u)) = 0$ $\forall i \in \overline{1, n-1}$.

Proof. The diagonalization of $\hat{\tau}_n(u)$ is performed recursively in several steps, utilizing the chain of embeddings $gl(n) \supset gl(n-1) \supset \cdots \supset gl(1)$. We will consider these steps separately in the corresponding subsubsections.

3.1.1. Step 1

Let us at first consider a subspace $V_0 \subset V$ consisting of the vectors $\mathbf{v}$ such that:

$$\hat{L}_{11}(u)\mathbf{v} = \Lambda_{11}(u)\mathbf{v}, \qquad \hat{L}_{k1}(u)\mathbf{v} = 0, \quad k > 1,$$

where $\Lambda_{11}(u)$ is a scalar function of u. Using the commutation relations in the algebra of the Lax operators it is easy to see that this subspace is invariant with respect to the action of the subalgebra of $gl(n-1)$-valued Lax operators generated by $\hat{L}_{kl}(u)$, $k, l > 1$. Moreover each of the vectors $\mathbf{v}$ in this subspace is obtained by the action of the elements $\hat{L}_{kl}(u)$, $k, l > 1$, $k < l$ on the vacuum vector $\boldsymbol{\Omega}$. Consideration of this subspace permit one to reduce a problem of the diagonalization of the generating functions $\hat{\tau}_n(u)$ of commutative integrals of $gl(n)$-valued Lax operators to the problem of the diagonalization of generating functions $\hat{\tau}_{n-1}(u)$, $\hat{t}_{n-1}(u)$ of the subalgebra of $gl(n-1)$-valued Lax operators.

Here we come to the next idea used in nested Bethe anzatz. Namely, in order to perform the Bethe anzatz correctly one has to perform a diagonalization on the next step of the recursive procedure not of $\hat{\tau}_{n-1}(u)$ acting in the space V_0, but of some function $\hat{\tau}^{(1)}_{n-1}(u)$ acting in the space $V_0 \otimes (\mathbb{C}^{n-1})^{\otimes M}$. For this purpose it will be convenient to consider also the action of $\hat{\tau}_n(u)$ lifted to the tensor product $V \otimes (\mathbb{C}^n)^{\otimes M}$ via the formula (23) below. Let us consider special vectors in $V \otimes (\mathbb{C}^n)^{\otimes M}$ of the form $\mathbf{V} = \sum_{i_1,\dots,i_M=2}^{n} v_{i_1 i_2 \dots i_M} e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_M}$ such that each $v_{i_1 i_2 \dots i_M}$ belongs to the space V_0 and e_i are basis vectors in the space $\mathbb{C}^n$ (vector-columns with unit on the place i and zeros everywhere else). These special vectors constitute the subspace $V_0 \otimes (\mathbb{C}^{n-1})^{\otimes M}$.

Let us now consider Bethe-type vectors of the following form:

$$\begin{aligned} \mathbf{V}(v_1, \dots, v_M) &= \hat{B}_1(v_1) \dots \hat{B}_M(v_M)\mathbf{V} \\ &= \sum_{i_1,\dots,i_M=2}^{n} \hat{L}_{1i_1}(v_1) \dots \hat{L}_{1i_M}(v_M) v_{i_1 i_2 \dots i_M}(e_1 \otimes \dots \otimes e_1), \end{aligned} \tag{22}$$

where the operators $\hat{B}_k(v_k)$ are defined as follows:

$$\hat{B}_k(v_k) \equiv 1_n \otimes \dots \otimes \hat{B}(v_k) \otimes \dots \otimes 1_n, \qquad \hat{B}(v_k) = \sum_{j=2}^{n} \hat{L}_{1j}(v) X_{1j}.$$

We will also introduce the following operator-valued matrices:

$$\hat{T}_{n,M}(u) \equiv \hat{\tau}_n(u) 1_n \otimes \dots \otimes 1_n \otimes \dots \otimes 1_n. \tag{23}$$

The Bethe-type vector (22) is identified with the following element of the space V:

$$\mathbf{v}(v_1, \dots, v_M) = \sum_{i_1,\dots,i_M=2}^{n} \hat{L}_{1i_1}(v_1) \dots \hat{L}_{1i_M}(v_M) v_{i_1 i_2 \dots i_M}.$$

It is evident that this identification is correct, the operators $\hat{T}_{n,M}(u)$ on $V \otimes (\mathbb{C}^n)^{\otimes M}$ and $\hat{\tau}_n(u)$ on V act in the same way and have the same spectrum.

Let us diagonalize the operator $\hat{T}_{n,M}(u)$ on $V \otimes (\mathbb{C}^n)^{\otimes M}$ using Bethe vectors (22):

$$\begin{aligned} \hat{T}_{n,M}(u)\mathbf{V}(v_1, \dots, v_M) &= \hat{B}_1(v_1) \dots \hat{B}_M(v_M) \hat{T}_{n,M}(u)\mathbf{V} \\ &\quad + \big[\hat{T}_{n,M}(u), \hat{B}_1(v_1) \dots \hat{B}_M(v_M)\big]\mathbf{V}. \end{aligned} \tag{24}$$

Let us calculate the first summand in this expression:

$$\hat{B}_1(v_1)\dots\hat{B}_M(v_M)\hat{T}_{n,M}(u)\mathbf{V}$$
$$= \hat{B}_1(v_1)\dots\hat{B}_M(v_M)\sum_{i_1,\dots,i_M=2}^{n}\hat{\tau}_n(u)(v_{i_1\dots i_M})(e_{i_1}\otimes\dots\otimes e_{i_M}).$$

For this purpose we will represent the generating function $\hat{\tau}_n(u)$ in the following form:

$$\hat{\tau}_n(u) = \frac{1}{2}\hat{L}_{11}(u)\hat{L}_{11}(u) + \hat{\tau}_{n-1}(u)$$
$$+ \frac{1}{2}\big(\partial_u + \big(r^0_{-,1}(u,u) + r^0_{+,1}(u,u)\big)\big)\big((n-1)\hat{L}_{11}(u) - \hat{t}_{n-1}(u)\big)$$
$$+ \sum_{j=2}^{n}\hat{L}_{1j}(u)\hat{L}_{j1}(u), \tag{25}$$

where $\hat{\tau}_{n-1}(u) \equiv \sum_{i,j=2}^{n}\hat{L}_{ij}(u)\hat{L}_{ji}(u)$. The formula (25) is obtained using the definition of $\hat{\tau}_n(u)$, the commutation relations in Lax algebra and condition (19a) with $i=1$.

Using the formula (25) and taking into attention that $v_{i_1\dots i_M} \in V_0$ we obtain:

$$\hat{B}_1(v_1)\dots\hat{B}_M(v_M)\hat{T}_{n,M}(u)\mathbf{V}$$
$$= \left(\frac{1}{2}\big(\Lambda_{11}(u)\big)^2 + \frac{(n-1)}{2}\big(\partial_u + \big(r^0_{-,1}(u,u) + r^0_{+,1}(u,u)\big)\big)\Lambda_{11}(u)\right)\mathbf{V}(v_1,\dots,v_M)$$
$$- \frac{1}{2}\hat{B}_1(v_1)\dots\hat{B}_M(v_M)\big(\partial_u + \big(r^0_{-,1}(u,u) + r^0_{+,1}(u,u)\big)\big)\hat{t}_{n-1}(u)\mathbf{V}$$
$$+ \hat{B}_1(v_1)\dots\hat{B}_M(v_M)\hat{\tau}_{n-1}(u)\mathbf{V}. \tag{26}$$

The second summand in the expression (24) is calculated using the following lemma:

Lemma 3.1. *Let the r-matrix under the consideration satisfy the condition (19a) with $i=1$:*

$$r_{j1}(u,v) = r_{-,1}(u,v), \qquad r_{1j}(u,v) = r_{+,1}(u,v) \quad \forall j \in \overline{2,n}. \tag{27}$$

Then the following commutation relations are valid:

$$\big[\hat{T}_{n,M}(u), \hat{B}_1(v_1)\dots\hat{B}_N(v_M)\big]$$
$$= \sum_{i=1}^{M} r_{-,1}(v_i,u)\hat{B}_1(v_1)\dots\check{\hat{B}}_i(v_i)\hat{B}_i(u)\dots\hat{B}_M(v_M)\mathrm{res}_{u=v_i}$$
$$\times\left(\frac{1}{2}\left(\hat{L}_{11}(u) - \sum_{k=1}^{M} r_{11}(v_k,u)\,\mathrm{Id}\right)^2\right.$$
$$\left. + \hat{\tau}^{(1)}_{n-1}(u) - \frac{n}{2}\sum_{j=1}^{M} r_{-,1}(v_j,u)r_{+,1}(v_j,u)\,\mathrm{Id}\right)$$
$$+ \hat{B}_1(v_1)\dots\hat{B}_N(v_M)\left(\frac{1}{2}\left(\hat{L}_{11}(u) - \sum_{k=1}^{M} r_{11}(v_k,u)\,\mathrm{Id}\right)^2\right.$$
$$\left. + \hat{\tau}^{(1)}_{n-1}(u) - \hat{L}^2_{11}(u) - \hat{\tau}_{n-1}(u) - \frac{n}{2}\sum_{j=1}^{M} r_{-,1}(v_j,u)r_{+,1}(v_j,u)\,\mathrm{Id}\right), \tag{28}$$

where "check" over the operator $\hat{B}_i(v_i)$ means that it is absent in the product,

$$\hat{\tau}^{(1)}_{n-1}(u) = \frac{1}{2}\operatorname{tr}_{n-1}\big(\hat{L}^{(1)}(u)\big)^2 \equiv \sum_{i,j=2}^{n} \hat{L}^{(1)}_{ij}(u)\hat{L}^{(1)}_{ji}(u)$$

and the operator $\hat{L}^{(1)}(u)$ is defined as follows:

$$\hat{L}^{(1)}(u) = \sum_{i,j=2}^{n} \hat{L}_{ij}(u)X_{ij} + \sum_{i,j=2}^{n}\sum_{k=1}^{M} r_{ij}(v_k, u)X^{(k)}_{ji}X_{ij},$$

and $X^{(k)}_{ji}$ denotes the operator that acts as X_{ij} in the k-th space of the tensor product of M spaces $\mathbb{C}^{n-1}$ and as unit operator in all other spaces $\mathbb{C}^{n-1}$ in this product.

(The lemma is proven by the direct calculation, using commutation relations in the Lax algebra, classical Yang–Baxter equation written in a component form, its differential consequences and condition (27). We do not present the explicit proof here due to its lengthy and technical character.)

Substituting the equality (28) in (24) and using the formula (26) we obtain the following action formula:

$$\begin{aligned}
&\hat{T}_{n,M}(u)\mathbf{V}(v_1, \dots, v_M) \\
&= \Bigg(\frac{1}{2}\Bigg(\Lambda_{11}(u) - \sum_{k=1}^{M} r_{11}(v_k, u)\Bigg)^2 + \frac{(n-1)}{2}\big(\partial_u + \big(r^0_{-,1}(u,u) + r^0_{+,1}(u,u)\big)\big)\Lambda_{11}(u) \\
&\quad - \frac{n}{2}\sum_{j=1}^{M} r_{-,1}(v_j, u) r_{+,1}(v_j, u)\Bigg)\mathbf{V}(v_1, \dots, v_M) \\
&\quad + \hat{B}_1(v_1)\dots\hat{B}_M(v_M)\big(\hat{\tau}^{(1)}_{n-1}(u) - \big(\partial_u + \big(r^0_{-,1}(u,u) + r^0_{+,1}(u,u)\big)\big)\hat{t}_{n-1}(u)\big)\mathbf{V} \\
&\quad + \sum_{i=1}^{M} r_{-,1}(v_i, u)\hat{B}_1(v_1)\dots\check{\hat{B}}_i(v_i)\hat{B}_i(u)\dots\hat{B}_M(v_M) \\
&\quad \times \operatorname{res}_{u=v_i}\Bigg(\frac{1}{2}\Bigg(\Lambda_{11}(u) - \sum_{k=1}^{M} r_{11}(v_k, u)\Bigg)^2 \operatorname{Id} \\
&\quad + \hat{\tau}^{(1)}_{n-1}(u) - \frac{n}{2}\sum_{j=1}^{M} r_{-,1}(v_j, u) r_{+,1}(v_j, u)\operatorname{Id}\Bigg)\mathbf{V}.
\end{aligned} \tag{29}$$

From this action formula it is evident, that the vector $\mathbf{V}(v_1, \dots, v_M)$ is an eigen-vector for $\hat{T}_{n,M}(u)$ if $\mathbf{V}$ is an eigen-vector for $\hat{\tau}^{(1)}_{n-1}(u)$ and $\hat{t}_{n-1}(u)$ with the eigen-values $\Lambda^{(1)}_{n-1}(u)$ and $c_{n-1}(u)$:

$$\hat{\tau}^{(1)}_{n-1}(u)\mathbf{V} = \Lambda^{(1)}_{n-1}(u)\mathbf{V}, \qquad \hat{t}_{n-1}(u)\mathbf{V} = c_{n-1}(u)\mathbf{V}$$

and the following Bethe equations are satisfied:

$$\Lambda_{11}(v_i) - \sum_{j=1; j\neq i}^{M} r_{11}(v_k, v_i)$$
$$- \left(r^0_{11}(v_i, v_i) - \frac{n}{2}\big(r^0_{-,1}(v_i, v_i) + r_{+,1}(v_i, v_i)\big)\right) + \mathrm{res}_{u=v_i} \Lambda^{(1)}_{n-1}(u) = 0. \tag{30}$$

In this case the eigen-value of $\hat{T}_{n,M}(u)$ and $\hat{\tau}_n(u)$ is

$$\Lambda_n(u) = \frac{1}{2}\left(\Lambda_{11}(u) - \sum_{k=1}^{M} r_{11}(v_k, u)\right)^2$$
$$+ \frac{1}{2}\big(\partial_u + \big(r^0_{-,1}(u,u) + r^0_{+,1}(u,u)\big)\big)\left(n\Lambda_{11}(u) - \sum_{i=1}^{n} \Lambda_{ii}(u)\right)$$
$$+ \Lambda^{(1)}_{n-1}(u) - \frac{n}{2}\sum_{j=1}^{M} r_{-,1}(v_j, u) r_{+,1}(v_j, u), \tag{31}$$

where we have used that the $\partial_u c_{n-1}(u) = \partial_u \sum_{i=2}^{n} \Lambda_{ii}(u)$ due to the fact that by the virtue of the condition (19b) $\hat{t}_{n-1}(u)$ is a Casimir operator of the $gl(n-1)$-valued Lax subalgebra, or by the virtue of the condition (19c) $\partial_u \hat{t}_{n-1}(u)$ is a Casimir operator of the $gl(n-1)$-valued Lax subalgebra and their eigen-values on the vector $v_{i_1\ldots i_M} \in V_0$ coincide with their eigen-values on the vacuum vector $\boldsymbol{\Omega}$.

Hence, we have reduced the problem of finding the spectrum of the generating function $\hat{\tau}_n(u)$ of the quantum integrals for $gl(n)$-valued Lax operators to the problem of finding the spectrum of $\hat{\tau}^{(1)}_{n-1}(u)$ of the generating function of the quantum integrals for the $gl(n-1)$-valued Lax operator possessing additional (in comparison to initial Lax operator) first order poles in the points v_i, $i \in \overline{1, M}$ that solve Eqs. (30).

3.1.2. Step 2

Let us now put in the above formulas (in particular in formula (31)): $v_i \equiv v_i^{(1)}$, $M \equiv M_1$ and make the next step in the recursive procedure of the nested Bethe anzatz. Let us assume that $n-1 > 1$ and write in the analogous way the corresponding formula for the spectrum of $\hat{\tau}^{(1)}_{n-1}(u)$:

$$\Lambda^{(1)}_{n-1}(u) = \frac{1}{2}\left(\Lambda^{(1)}_{22}(u) - \sum_{k=1}^{M_2} r_{22}\big(v_k^{(2)}, u\big)\right)^2$$
$$+ \frac{1}{2}\big(\partial_u + \big(r^0_{-,2}(u,u) + r^0_{+,2}(u,u)\big)\big)\left((n-1)\Lambda^{(1)}_{22}(u) - \sum_{i=2}^{n} \Lambda^{(1)}_{ii}(u)\right)$$
$$+ \Lambda^{(2)}_{n-2}(u) - \frac{n-1}{2}\sum_{j=1}^{M_2} r_{-,2}\big(v_j^{(2)}, u\big) r_{+,2}\big(v_j^{(2)}, u\big), \tag{32}$$

where we have used the conditions (19a) with $i = 2$ imposed on the matrix elements of the r-matrix.

Now we have to define how $\Lambda^{(1)}_{22}(u), \Lambda^{(1)}_{33}(u), \ldots, \Lambda^{(1)}_{nn}(u)$ are looking like in the terms of the original eigen-values $\Lambda_{22}(u), \Lambda_{33}(u), \ldots, \Lambda_{nn}(u)$. For this purpose we will consider in more details this step of the nested Bethe anzatz.

Let us consider the subspace $V_0^{(1)}$ (in the space $V^{(1)} = V_0 \otimes (\mathbb{C}^{n-1})^{\otimes M_1}$ of representation of the subalgebra $gl(n-1)$) consisting of the vectors $\mathbf{v}^{(1)}$ such that:

$$\hat{L}_{22}^{(1)}(u)\mathbf{v}^{(1)} = \Lambda_{22}^{(1)}(u)\mathbf{v}^{(1)}, \qquad \hat{L}_{k2}^{(1)}(u)\mathbf{v}^{(1)} = 0, \quad k > 2,$$

where

$$\hat{L}^{(1)}(u) = \sum_{i,j=2}^{n} \hat{L}_{ij}^{(1)}(u)X_{ij} = \sum_{i,j=2}^{n} \left(\hat{L}_{ij}(u) + \sum_{k=1}^{M_1} r_{ij}\big(v_k^{(1)}, u\big)X_{ji}^{(k)} \right) X_{ij}$$

is the new $gl(n-1)$-valued Lax operator acting in the space $V^{(1)} = V_0 \otimes (\mathbb{C}^{n-1})^{\otimes M_1}$. From the condition $\hat{L}_{k2}^{(1)}(u)\mathbf{v}^{(1)} = 0$ it follows that vectors $\mathbf{v}^{(1)}$ have the following form:

$$\mathbf{v}^{(1)} = \mathbf{v} \otimes e_2 \otimes e_2 \otimes \cdots \otimes e_2.$$

Indeed, due to the fact that $\hat{L}(u)$ has no poles at the points $u = v_k^{(1)}$, the condition $\hat{L}_{k2}^{(1)}(u)\mathbf{v}^{(1)} = 0$ implies, in particular, that $X_{2k}^{(l)}\mathbf{v}^{(1)} = 0$, for $k > 2$ and, hence, $\mathbf{v}^{(1)} = \mathbf{v} \otimes e_2 \otimes e_2 \otimes \cdots \otimes e_2$, where $\mathbf{v}$ is such that $\hat{L}_{k2}(u)\mathbf{v} = 0$.

Remark 8. We will not repeat here all the lengthy procedure of diagonalization of $\widehat{\tau}_{n-1}^{(1)}(u)$ which is essentially the same as for the case of $\widehat{\tau}_n(u)$. Let us only remark that the Bethe vectors, on which $\widehat{\tau}_{n-1}^{(1)}(u)$ is diagonalized, are constructed, as in the previous subsection, using the following formula:

$$\mathbf{v}^{(1)}\big(v_1^{(2)}, \dots, v_{M_2}^{(2)}\big) = \sum_{i_1,\dots,i_{M_2}=3}^{n} \hat{L}_{2i_1}^{(1)}\big(v_1^{(2)}\big) \dots \hat{L}_{2i_M}^{(1)}\big(v_{M_2}^{(2)}\big) v_{i_1 i_2 \dots i_{M_2}}^{(1)},$$

where $v_{i_1 i_2 \dots i_{M_2}}^{(1)} \in V_0^{(1)}$ and $\hat{L}_{2j}^{(1)}(u) = \hat{L}_{2j}(u) + \sum_{k=1}^{M_1} r_{2j}(v_k^{(1)}, u)X_{j2}^{(k)}$.

Let us use the explicit form of the vectors $\mathbf{v}^{(1)}$ in order to calculate the explicit form of $\Lambda_{22}^{(1)}(u)$ and $\sum_{i=2}^{n} \Lambda_{ii}^{(1)}(u)$. Using that $\hat{L}_{22}^{(1)}(u) = \hat{L}_{22}(u) + \sum_{k=1}^{M_1} r_{22}(v_k^{(1)}, u)X_{22}^{(k)}$ we obtain:

$$\Lambda_{22}^{(1)}(u) = \Lambda_{22}(u) + \sum_{k=1}^{M_1} r_{22}\big(v_k^{(1)}, u\big).$$

Due to the fact that $\sum_{i=2}^{n} \Lambda_{ii}^{(1)}(u)$ appeared in the formula for spectrum as an eigen-value of the linear Casimir operator of $gl(n-1)$ we obtain that it has the same value on any vector of $V^1 = V_0 \otimes (\mathbb{C}^{n-1})^{\otimes M_1}$ as on the "vacuum vector" $\boldsymbol{\Omega} \otimes e_2 \otimes e_2 \otimes \cdots \otimes e_2$. This value is equal to:

$$\sum_{i=2}^{n} \Lambda_{ii}^{(1)}(u) = \sum_{i=2}^{n} \Lambda_{ii}(u) + \sum_{k=1}^{M_1} r_{22}\big(v_k^{(1)}, u\big).$$

These considerations permit us to write down the corresponding formula for the spectrum of $\hat{\tau}_{n-1}^{(1)}(u)$ more explicitly:

$$\Lambda^{(1)}_{n-1}(u) = \frac{1}{2}\left(\Lambda_{22}(u) + \sum_{k=1}^{M_1} r_{22}\left(v_k^{(1)}, u\right) - \sum_{k=1}^{M_2} r_{22}\left(v_k^{(2)}, u\right)\right)^2$$
$$+ \frac{1}{2}\left(\partial_u + \left(r^0_{-,2}(u,u) + r^0_{+,2}(u,u)\right)\right)$$
$$\times \left((n-1)\left(\Lambda_{22}(u) + \sum_{k=1}^{M_1} r_{22}\left(v_k^{(1)}, u\right)\right) - \left(\sum_{i=2}^{n} \Lambda_{ii}(u) + \sum_{k=1}^{M_1} r_{22}\left(v_k^{(1)}, u\right)\right)\right)$$
$$- \frac{n-1}{2} \sum_{j=1}^{M_2} r_{-,2}\left(v_j^{(2)}, u\right) r_{+,2}\left(v_j^{(2)}, u\right) + \Lambda^{(2)}_{n-2}(u). \tag{33}$$

Now we can recalculate the first set of the Bethe equations (30). They acquire the form:

$$\Lambda_{11}\left(v_i^{(1)}\right) - \Lambda_{22}\left(v_i^{(1)}\right) - \left(r^0_{11}(v_i, v_i) + r^0_{22}(v_i, v_i)\right) + \frac{n}{2}\left(r^0_{-,1}(v_i, v_i) + r_{+,1}(v_i, v_i)\right)$$
$$- \frac{n-2}{2}\left(r^0_{-,2}(v_i, v_i) + r_{+,2}(v_i, v_i)\right)$$
$$= \sum_{j=1;\, j\neq i}^{M} \left(r_{11}\left(v_j^{(1)}, v_i^{(1)}\right) + r_{22}\left(v_j^{(1)}, v_i^{(1)}\right)\right) - \sum_{j=1}^{M_2} r_{22}\left(v_j^{(2)}, v_i^{(1)}\right), \tag{34}$$

because, as it will follow from the results of the next subsubsection $\operatorname{res}_{u=v_i^{(1)}} \Lambda^{(2)}_{n-2}(u) = 0$. Using the recursive character of our procedure we write in analogous way the second set of Bethe equations:

$$\Lambda^{(1)}_{22}\left(v_i^{(2)}\right) - \sum_{j=1;\, j\neq i}^{M_2} r_{22}\left(v_j^{(2)}, v_i^{(2)}\right)$$
$$- \left(r^0_{22}\left(v_i^{(2)}, v_i^{(2)}\right) - \frac{n-1}{2}\left(r^0_{-,2}\left(v_i^{(2)}, v_i^{(2)}\right) + r_{+,2}\left(v_i^{(2)}, v_i^{(2)}\right)\right)\right)$$
$$+ \operatorname{res}_{u=v_i^{(2)}} \Lambda^{(2)}_{n-2}(u) = 0. \tag{35}$$

Now in order to obtain all formulas in final form we have to calculate explicitly $\Lambda^{(2)}_{n-2}(u)$, $\Lambda^{(3)}_{n-3}(u)$, etc., making all the subsequent steps in the nested Bethe ansatz.

3.1.3. Next steps and end of the proof

Let us now consider next steps in the algebraic Bethe ansatz. Let us consider formula for the spectrum of second order generating function of the quantum integrals of $gl(n-k)$ which are obtained on the $k+1$-th step of the recursive procedure:

$$\Lambda^{(k)}_{n-k}(u) = \frac{1}{2}\left(\Lambda^{(k)}_{k+1k+1}(u) - \sum_{j=1}^{M_{k+1}} r_{k+1k+1}\left(v_j^{(k+1)}, u\right)\right)^2$$
$$+ \frac{1}{2}\left(\partial_u + \left(r^0_{-,k+1}(u,u) + r^0_{+,k+1}(u,u)\right)\right)$$
$$\times \left((n-k)\Lambda^{(k)}_{k+1k+1}(u) - \sum_{i=k+1}^{n} \Lambda^{(k)}_{ii}(u)\right)$$

$$-\frac{n-k}{2}\sum_{j=1}^{M_{k+1}} r_{-,k+1}\big(v_j^{(k+1)},u\big)r_{+,k+1}\big(v_j^{(k+1)},u\big)$$
$$+\Lambda^{(k+1)}_{n-k-1}(u), \tag{36}$$

where we assumed that $k \geq 0$, $n-k \geq 2$, $\Lambda^{(k)}_{ii}(u)$ denotes eigen-value of $\hat{L}^{(k)}_{ii}(u)$ and $\hat{L}^{(k)}(u)$ is a "modified" Lax operator that has obtained additional poles after k times application of the described in previous subsubsections recursive procedure. In particular, $\Lambda^{(0)}_{ii}(u) \equiv \Lambda_{ii}(u)$, etc.

Using the same arguments as in the previous subsubsection we obtain the following explicit expressions for $\Lambda^{(k)}_{ii}(u)$:

$$\Lambda^{(k)}_{k+1k+1}(u) = \Lambda_{k+1k+1}(u) + \sum_{i=1}^{M_k} r_{k+1k+1}\big(v_i^{(k)},u\big),$$
$$\sum_{i=k+1}^{n} \Lambda^{(k)}_{ii}(u) = \sum_{i=k+1}^{n} \Lambda_{ii}(u) + \sum_{i=1}^{M_k} r_{k+1k+1}\big(v_i^{(k)},u\big).$$

In order to make a final calculation of the spectrum we have only to use the evident formula for the spectrum of the generating function of the quadratic integrals of the last $gl(1)$ subalgebra in the chain:

$$\Lambda^{(n-1)}_1(u) = \frac{1}{2}\left(\Lambda_{nn}(u) + \sum_{j=1}^{M_{n-1}} r_{nn}\big(v_j^{(n-1)},u\big)\right)^2 \tag{37}$$

Using all these and applying $n-1$ times formula (36) we finally obtain explicit expression for the spectrum of the generating function $\hat{\tau}_n(u)$:

$$2\Lambda_n(u) = \sum_{k=1}^{n}\left(\Lambda_{kk}(u) + \sum_{j=1}^{M_{k-1}} r_{kk}\big(v_j^{(k-1)},u\big) - \sum_{j=1}^{M_k} r_{kk}\big(v_j^{(k)},u\big)\right)^2$$
$$+\sum_{k=1}^{n}\big(\partial_u + \big(r^0_{-,k}(u,u) + r^0_{+,k}(u,u)\big)\big)$$
$$\times\left((n-k)\Lambda_{kk}(u) - \sum_{i=k+1}^{n}\Lambda_{ii}(u) + (n-k)\sum_{j=1}^{M_{k-1}} r_{kk}\big(v_j^{(k-1)},u\big)\right)$$
$$-\sum_{k=1}^{n}(n-k+1)\sum_{j=1}^{M_k} r_{-,k}\big(v_j^{(k)},u\big)r_{+,k}\big(v_j^{(k)},u\big).$$

It is left to calculate explicit expressions for all sets of Bethe equations, that are satisfied by the rapidities $v_i^{(k)}$ entering into the above formula for the spectrum $\hat{\tau}_n(u)$.

The $(k+1)$-th set of the Bethe equations reads as follows:

$$\Lambda^{(k)}_{k+1k+1}\big(v_i^{(k+1)}\big) - \sum_{j=1;\, j\neq i}^{M_{k+1}} r_{k+1k+1}\big(v_j^{(k+1)},v_i^{(k+1)}\big)$$
$$-\left(r^0_{k+1k+1}\big(v_i^{(k+1)},v_i^{(k+1)}\big) - \frac{n-k}{2}\right.$$

$$\times \Big(r^0_{-,k+1}\big(v_i^{(k+1)}, v_i^{(k+1)}\big) + r_{+,k+1}\big(v_i^{(k+1)}, v_i^{(k+1)}\big)\big)\Big)$$
$$+ \operatorname{res}_{u=v_i^{(k+1)}} \Lambda^{(k+1)}_{n-k-1}(u) = 0. \tag{38}$$

Taking into account formula (36) with $k \to k+1$ we obtain that

$$\begin{aligned}
&\operatorname{res}_{u=v_i^{(k+1)}} \Lambda^{(k+1)}_{n-(k+1)}(u) \\
&= -\Lambda_{k+2k+2}\big(v_i^{(k+1)}\big) - \sum_{j=1; j\neq i}^{M_{k+1}} r_{k+2k+2}\big(v_j^{(k+1)}, v_i^{(k+1)}\big) - r^0_{k+2k+2}\big(v_i^{(k+1)}, v_i^{(k+1)}\big) \\
&\quad + \sum_{j=1}^{M_{k+2}} r_{k+2k+2}\big(v_j^{(k+2)}, v_i^{(k+1)}\big) \\
&\quad + \frac{n-k-2}{2}\big(r^0_{-,k+2}\big(v_i^{(k+1)}, v_i^{(k+1)}\big) + r^0_{+,k+2}\big(v_i^{(k+1)}, v_i^{(k+1)}\big)\big).
\end{aligned}$$

At last, due to the said above:

$$\Lambda^{(k)}_{k+1k+1}\big(v_i^{(k+1)}\big) = \Lambda_{k+1k+1}\big(v_i^{(k+1)}\big) + \sum_{j=1}^{M_k} r_{k+1k+1}\big(v_j^{(k)}, v_i^{(k+1)}\big). \tag{39}$$

Hence we obtain the following sets of the Bethe equations for $k \in \overline{1, n-3}$:

$$\begin{aligned}
&\Lambda_{k+1k+1}\big(v_i^{(k+1)}\big) - \Lambda_{k+2k+2}\big(v_i^{(k+1)}\big) - \big(r^0_{k+1k+1}\big(v_i^{(k+1)}, v_i^{(k+1)}\big) \\
&\quad + r^0_{k+2k+2}\big(v_i^{(k+1)}, v_i^{(k+1)}\big)\big) + \frac{n-k}{2} \\
&\quad \times \big(r^0_{-,k+1}\big(v_i^{(k+1)}, v_i^{(k+1)}\big) + r_{+,k+1}\big(v_i^{(k+1)}, v_i^{(k+1)}\big)\big) \\
&\quad - \frac{n-k-2}{2}\big(r^0_{-,k+2}\big(v_i^{(k+1)}, v_i^{(k+1)}\big) + r^0_{+,k+2}\big(v_i^{(k+1)}, v_i^{(k+1)}\big)\big) \\
&= \sum_{j=1; j\neq i}^{M_{k+1}} \big(r_{k+1k+1}\big(v_j^{(k+1)}, v_i^{(k+1)}\big) + r_{k+2k+2}\big(v_j^{(k+1)}, v_i^{(k+1)}\big)\big) \\
&\quad - \sum_{j=1}^{M_k} r_{k+1k+1}\big(v_j^{(k)}, v_i^{(k+1)}\big) - \sum_{j=1}^{M_{k+2}} r_{k+2k+2}\big(v_j^{(k+2)}, v_i^{(k+1)}\big).
\end{aligned} \tag{40}$$

They go together with the set of Eqs. (34) and the following set of equations:

$$\begin{aligned}
&\Lambda_{n-1n-1}\big(v_i^{(n-1)}\big) - \Lambda_{nn}\big(v_i^{(n-1)}\big) - \big(r^0_{n-1n-1}\big(v_i^{(n-1)}, v_i^{(n-1)}\big) + r^0_{nn}\big(v_i^{(n-1)}, v_i^{(n-1)}\big)\big) \\
&\quad + r^0_{-,n-1}\big(v_i^{(n-1)}, v_i^{(n-1)}\big) + r_{+,n-1}\big(v_i^{(n-1)}, v_i^{(n-1)}\big) \\
&= \sum_{j=1; j\neq i}^{M_{n-1}} \big(r_{n-1n-1}\big(v_j^{(n-1)}, v_i^{(n-1)}\big) + r_{nn}\big(v_j^{(n-1)}, v_i^{(n-1)}\big)\big) \\
&\quad - \sum_{j=1}^{M_{n-2}} r_{n-1n-1}\big(v_j^{(n-2)}, v_i^{(n-1)}\big),
\end{aligned} \tag{41}$$

which are consequences of Eqs. (38), (39) calculated for $k = n - 2$ and equality (37).

Theorem is proven. □

3.2. Spectrum of the linear integrals

Let us describe the additional linear integrals that are connected with the geometric symmetry (Cartan-invariance) of the r-matrix. In more details, let us assume that among the quantum dynamical variables there exist quantum operators $\hat{M}_{X_{kk}}$, $k \in \overline{1,n}$ such that the following condition is satisfied [34]:

$$\left[\hat{M}_{X_{kk}}, \hat{L}_{ij}(u)\right] = (\delta_{ki} - \delta_{kj})\hat{L}_{ij}(u). \tag{42}$$

In this case it is possible to show that $[\hat{M}_{X_{kk}}, \hat{M}_{X_{ll}}] = 0$, $k, l \in \overline{1,n}$ and, moreover,

$$\left[\hat{M}_{X_{kk}}, \operatorname{tr}\hat{L}(u)\right] = 0, \qquad \left[\hat{M}_{X_{kk}}, \operatorname{tr}\hat{L}^2(u)\right] = 0, \quad k \in \overline{1,n},$$

i.e. there exist additional commuting integrals diagonalized together with $\hat{\tau}_n(u)$.

The following proposition holds true:

Proposition 3.1. *The action of the additional linear integrals $\hat{M}_{X_{ii}}$, $i \in \overline{1,n}$ on the "nested" Bethe vectors $\mathbf{V}(v^{(1)}_{m_1}, \ldots, v^{(n-1)}_{m_{n-1}})$ has the following diagonal form:*

$$\hat{M}_{X_{ii}}\mathbf{V}\left(v^{(1)}_{m_1}, \ldots, v^{(n-1)}_{m_{n-1}}\right) = m_i(M_1, \ldots, M_{n-1})\mathbf{V}\left(v^{(1)}_{m_1}, \ldots, v^{(n-1)}_{m_{n-1}}\right), \quad \textit{where}$$
$$m_i(M_1, \ldots, M_{n-1}) = m_{ii} + M_{i-1} - M_i,$$

Here m_{ii} is an eigen-value of $\hat{M}_{X_{ii}}$ on the vacuum vector $\boldsymbol{\Omega}$:

$$\hat{M}_{X_{ii}}\boldsymbol{\Omega} = m_{ii}\boldsymbol{\Omega}$$

and, by the very definition, $M_0 = M_n = 0$.

(The proof of the proposition is the same as in the case of the rational r-matrix [34].)

At last, for the sake of completeness, let us calculate the spectrum of the linear integrals obtained from the generating function $\hat{t}_n(u)$.

The following proposition holds true:

Proposition 3.2. *The action of the generating function of the linear integrals $\hat{t}_n(u)$ on the "nested" Bethe vectors $\mathbf{V}(v^{(1)}_{m_1}, \ldots, v^{(n-1)}_{m_{n-1}})$ has the following diagonal form:*

$$\hat{t}_n(u)\mathbf{V}\left(v^{(1)}_{m_1}, \ldots, v^{(n-1)}_{m_{n-1}}\right) = \sum_{k=1}^{n}\left(\Lambda_{kk}(u) + \sum_{i=1}^{M_{k-1}} r_{kk}\left(v^{(k-1)}_i, u\right) - \sum_{i=1}^{M_k} r_{kk}\left(v^{(k)}_i, u\right)\right)\mathbf{V}\left(v^{(1)}_{m_1}, \ldots, v^{(n-1)}_{m_{n-1}}\right).$$

Idea of the proof. The proposition is proven recursively with the help of the following action formula obtained using the commutation relations in the Lax algebra and the definitions of the operators $\hat{t}_n(u)$, $\hat{B}_i(v^{(1)}_i)$:

$$\hat{t}_n(u)\hat{B}_1\big(v_1^{(1)}\big)\dots\hat{B}_{M_1}\big(v_{M_1}^{(1)}\big)=\hat{B}_1\big(v_1^{(1)}\big)\dots\hat{B}_M\big(v_{M_1}^{(1)}\big)\hat{t}_{n-1}^{(1)}(u)$$
$$+\hat{B}_1\big(v_1^{(1)}\big)\dots\hat{B}_M\big(v_{M_1}^{(1)}\big)\left(\hat{L}_{11}(u)-\sum_{i=1}^{n}r_{11}\big(v_i^{(1)},u\big)\right),$$

where $\hat{t}_{n-1}^{(1)}(u)=\sum_{k=2}^{n}\hat{L}_{kk}^{(1)}(u)$ and "modified" $gl(n-1)$-valued Lax operator $\hat{L}^{(1)}(u)$ is defined as in the previous subsection, i.e.: $L_{kl}^{(1)}(u)=\hat{L}_{kl}(u)+\sum_{i=1}^{M_1}r_{kl}(v_i^{(1)},u)X_{lk}^{(i)}$, $k,l\in\overline{2,n}$. □

Remark 9. The operators $\hat{t}_n(u)$ and $\hat{M}_{X_{ii}}$ in the above propositions are understand to be lifted onto the space $V\otimes(\mathbb{C}^n)^{\otimes M_1}\otimes\dots\otimes(\mathbb{C}^n)^{\otimes M_{n-1}}$ in the same simple way as the generating function $\hat{\tau}_n(u)$ was lifted.

Remark 10. In the case of the r-matrices satisfying the conditions (19b) the action formula of $\hat{t}_n(u)$ is simplified to have the following form:

$$\hat{t}_n(u)\mathbf{V}\big(v_{m_1}^{(1)},\dots,v_{m_{n-1}}^{(n-1)}\big)$$
$$=\left(\sum_{k=1}^{n}\Lambda_{kk}(u)+\sum_{i=1}^{M_1}r_{22}\big(v_i^{(1)},u\big)-\sum_{i=1}^{M_1}r_{11}\big(v_i^{(1)},u\big)\right)\mathbf{V}\big(v_{m_1}^{(1)},\dots,v_{m_{n-1}}^{(n-1)}\big).$$

3.3. Case of the Gaudin models

3.3.1. Spectrum of the generalized Gaudin Hamiltonians

Let us specify the obtained answers for spectrum and Bethe equations in the case of the generalized Gaudin models with and without an external magnetic field. By other words, let us calculate the spectrum of the corresponding Gaudin-type Hamiltonians. In order to do this it will be necessary to specify the explicit form of the eigen-values $\Lambda_{ii}(u)$.

In more details: in the cases of Gaudin-type models Lax algebra coincides with the direct sum of N-copies of the Lie algebra $gl(n)$. A space of an irreducible representation of the algebra $gl(n)^{\oplus N}$ has the form: $V=(\bigotimes_{l=1}^{N}V^{\lambda^{(l)}})$, where $V^{\lambda^{(l)}}$ is the space of an irreducible representation of the l-th copy of $gl(n)$ labeled by the highest weights $\lambda^{(l)}=(\lambda_1^{(l)},\dots,\lambda_n^{(l)})$, $l\in\overline{1,N}$. In the representation space V there exists the highest weight vector $\boldsymbol{\Omega}$ such that:

$$\hat{S}_{ii}^{(l)}\boldsymbol{\Omega}=\lambda_i^{(l)}\Omega,\qquad \hat{S}_{ij}^{(l)}\boldsymbol{\Omega}=0,\quad i,j\in\overline{1,n},\ i<j;\ l\in\overline{1,N}.$$

Using the definition of the Lax operators (14)–(15) it is easy to see that $\boldsymbol{\Omega}$ coincides with the vacuum vector for the corresponding representation of the Lax algebra.

The following corollary of Theorem 3.1 holds true:

Corollary 3.1. *Let the r-matrix $r(u,v)$ satisfy the conditions (4), (19a) and the condition (19b) or (19c). Then the spectrum of the generalized Gaudin Hamiltonians (16) and the generalized Gaudin Hamiltonians in an external magnetic field (17) has the form:*

$$h_l=h_l^0+\sum_{k=1}^{n}\lambda_k^{(l)}\left(\sum_{j=1}^{M_{k-1}}r_{kk}\big(v_j^{(k-1)},\nu_l\big)-\sum_{j=1}^{M_k}r_{kk}\big(v_j^{(k)},\nu_l\big)\right),\tag{43}$$

where h_l^0 is an eigen-value of these Hamiltonians on the vacuum vector $\boldsymbol{\Omega}$:

$$
\begin{aligned}
h_l^0 = {} & \sum_{k=1}^{n}\left(\sum_{m=1,m\neq l}^{N} r_{kk}(\nu_m,\nu_l)\lambda_k^{(m)}\lambda_k^{(l)} + r_{kk}^0(\nu_l,\nu_l)\big(\lambda_k^{(l)}\big)^2\right) \\
& + \frac{1}{2}\sum_{k=1}^{n}\left((n-k)\lambda_k^{(l)} - \sum_{s=k+1}^{n}\lambda_s^{(l)}\right)\big(r_{-,k}^0(\nu_l,\nu_l) + r_{+,k}^0(\nu_l,\nu_l)\big) \\
& + \sum_{k=1}^{n} c_{kk}(\nu_l)\lambda_k^{(l)},
\end{aligned}
\tag{44}
$$

$c_{kk}(\nu_l)$ are the components of the shift element — external magnetic field and the "rapidities" $v_k^{(i)}$, $k\in\overline{1,M_i}$, $i\in\overline{1,n-1}$ satisfy the following Bethe equations:

$$
\begin{aligned}
& \sum_{m=1}^{N}\lambda_1^{(m)} r_{11}\big(\nu_m, v_i^{(1)}\big) - \sum_{m=1}^{N}\lambda_2^{(m)} r_{22}\big(\nu_m, v_i^{(1)}\big) + \big(c_{11}\big(v_i^{(1)}\big) - c_{22}\big(v_i^{(1)}\big)\big) \\
& \quad - \big(r_{11}^0\big(v_i^{(1)}, v_i^{(1)}\big) + r_{22}^0\big(v_i^{(1)}, v_i^{(1)}\big)\big) \\
& \quad + \frac{n}{2}\big(r_{-,1}^0\big(v_i^{(1)}, v_i^{(1)}\big) + r_{+,1}^0\big(v_i^{(1)}, v_i^{(1)}\big)\big) - \frac{n-2}{2}\big(r_{-,2}^0\big(v_i^{(1)}, v_i^{(1)}\big) + r_{+,2}^0\big(v_i^{(1)}, v_i^{(1)}\big)\big) \\
& \quad = \sum_{j=1;\, j\neq i}^{M_1}\big(r_{11}\big(v_j^{(1)}, v_i^{(1)}\big) + r_{22}\big(v_j^{(1)}, v_i^{(1)}\big)\big) - \sum_{j=1}^{M_2} r_{22}\big(v_j^{(2)}, v_i^{(1)}\big),
\end{aligned}
\tag{45a}
$$

$$
\begin{aligned}
& \sum_{m=1}^{N}\lambda_{k+1}^{(m)} r_{k+1k+1}\big(\nu_m, v_i^{(k+1)}\big) - \sum_{m=1}^{N}\lambda_{k+2}^{(m)} r_{k+2k+2}\big(\nu_m, v_i^{(k+1)}\big) \\
& \quad + \big(c_{k+1k+1}\big(v_i^{(k+1)}\big) - c_{k+2k+2}\big(v_i^{(k+1)}\big)\big) \\
& \quad - \big(r_{k+1k+1}^0\big(v_i^{(k+1)}, v_i^{(k+1)}\big) + r_{k+2k+2}^0\big(v_i^{(k+1)}, v_i^{(k+1)}\big)\big) \\
& \quad + \frac{n-k}{2}\big(r_{-,k+1}^0\big(v_i^{(k+1)}, v_i^{(k+1)}\big) + r_{+,k+1}^0\big(v_i^{(k+1)}, v_i^{(k+1)}\big)\big) \\
& \quad - \frac{n-k-2}{2}\big(r_{-,k+2}^0\big(v_i^{(k+1)}, v_i^{(k+1)}\big) + r_{+,k+2}^0\big(v_i^{(k+1)}, v_i^{(k+1)}\big)\big) \\
& \quad = \sum_{j=1;\, j\neq i}^{M_{k+1}}\big(r_{k+1k+1}\big(v_j^{(k+1)}, v_i^{(k+1)}\big) + r_{k+2k+2}\big(v_j^{(k+1)}, v_i^{(k+1)}\big)\big) \\
& \quad - \sum_{j=1}^{M_k} r_{k+1k+1}\big(v_j^{(k)}, v_i^{(k+1)}\big) - \sum_{j=1}^{M_{k+2}} r_{k+2k+2}\big(v_j^{(k+2)}, v_i^{(k+1)}\big), \\
& \quad k\in\overline{1,n-3},
\end{aligned}
\tag{45b}
$$

$$
\begin{aligned}
& \sum_{m=1}^{N}\lambda_{n-1}^{(m)} r_{n-1n-1}\big(\nu_m, v_i^{(n-1)}\big) - \sum_{m=1}^{N}\lambda_{n}^{(m)} r_{nn}\big(\nu_m, v_i^{(n-1)}\big) \\
& \quad + \big(c_{n-1n-1}\big(v_i^{(n-1)}\big) - c_{nn}\big(v_i^{(n-1)}\big)\big) \\
& \quad - \big(r_{n-1n-1}^0\big(v_i^{(n-1)}, v_i^{(n-1)}\big) + r_{nn}^0\big(v_i^{(n-1)}, v_i^{(n-1)}\big)\big)
\end{aligned}
$$

$$
\begin{aligned}
&+r^0_{-,n-1}\big(v_i^{(n-1)}, v_i^{(n-1)}\big)+r^0_{+,n-1}\big(v_i^{(n-1)}, v_i^{(n-1)}\big)\\
&=\sum_{j=1;\, j\neq i}^{M_{n-1}}\big(r_{n-1n-1}\big(v_j^{(n-1)}, v_i^{(n-1)}\big)+r_{nn}\big(v_j^{(n-1)}, v_i^{(n-1)}\big)\big)\\
&-\sum_{j=1}^{M_{n-2}} r_{n-1n-1}\big(v_j^{(n-2)}, v_i^{(n-1)}\big).
\end{aligned}
\tag{45c}
$$

Remark 11. Observe that in the case of the generalized Gaudin Hamiltonians without external magnetic field (16) it is necessary to put $c_{kk}(u)=0$, $\forall k\in\overline{1,n}$ in the formulas (44) and (45).

Proof. The proof is achieved by the direct calculation, using the formulas (21), taking into account that in the case of the generalized Gaudin models $\Lambda_{kk}(u)=\sum_{m=1}^{N}\lambda_k^{(m)} r_{kk}(\nu_m,u)$, in the case of the generalized Gaudin models in an external magnetic field $\Lambda_{kk}(u)=\sum_{m=1}^{N}\lambda_k^{(m)} r_{kk}(\nu_m,u)+c_{kk}(u)$ and to take the residue in the point $u=\nu_l$ in the formula (20). □

3.3.2. Spectrum of the linear integrals

Let us describe the additional linear integrals that are important for the applications in Gaudin-type models. These will be the so-called "global spin operators".

In more details, using the commutation relations of the Lie algebra $gl^{\oplus N}(n)$ for the case of the diagonal r-matrices and diagonal shift elements it is possible to show that the following linear operators:

$$
\hat{M}_{X_{kk}}=\sum_{l=1}^{N} S_{kk}^{(l)}
$$

satisfy the condition (42) and hence are additional commuting integrals for the generalized Gaudin models with and without external magnetic field.

The following corollary of the Theorem 3.1 holds true:

Corollary 3.2. *The eigen-values of the additional linear integrals $\hat{M}_{X_{ii}}$, $i\in\overline{1,n}$ on the "nested" Bethe vectors $\mathbf{V}(v_{m_1}^{(1)},\dots,v_{m_{n-1}}^{(n-1)})$ has the following form:*

$$
\begin{aligned}
&\hat{M}_{X_{ii}}\mathbf{V}\big(v_{m_1}^{(1)},\dots,v_{m_{n-1}}^{(n-1)}\big)=m_i(M_1,\dots,M_{n-1})\mathbf{V}\big(v_{m_1}^{(1)},\dots,v_{m_{n-1}}^{(n-1)}\big),\quad \textit{where}\\
&m_i(M_1,\dots,M_{n-1})=\sum_{l=1}^{N}\lambda_i^{(l)}+M_{i-1}-M_i,
\end{aligned}
$$

and by the very definition, $M_0=M_n=0$.

At last, for the sake of completeness, let us calculate the spectrum of the linear integrals obtained from the generating function $\hat{t}_n(u)$.

The following proposition holds true:

Proposition 3.3. *The eigen-values of the generating function of the linear integrals $\hat{t}_n(u)$ on the "nested" Bethe vectors $\mathbf{V}(v_{m_1}^{(1)},\dots,v_{m_{n-1}}^{(n-1)})$ has the following form:*

$$\lambda_n(u) = \sum_{k=1}^{n} \left(\sum_{m=1}^{N} \lambda_k^{(m)} r_{kk}(\nu_m, u) + c_{kk}(u) + \sum_{i=1}^{M_{k-1}} r_{kk}\left(v_i^{(k-1)}, u\right) - \sum_{i=1}^{M_k} r_{kk}\left(v_i^{(k)}, u\right) \right). \tag{46}$$

4. Examples

In this section we will consider three classes of examples of the diagonal non-skew-symmetric classical r-matrices satisfying the conditions of Theorem 3.1, the corresponding Gaudin-type Hamiltonians, their spectrum and Bethe equations.

4.1. *Shifted rational r-matrices and Bethe ansatz*

4.1.1. *"Shifted" rational r-matrices*

Let us consider classical r-matrix of the following form:

$$r^{12}(u, v) = \frac{\Omega^{12}}{u - v} + c^{12}, \tag{47}$$

where c^{12} is the constant $gl(n) \otimes gl(n)$-valued solution of the generalized classical Yang–Baxter equation:

$$\left[c^{12}, c^{13}\right] = \left[c^{23}, c^{12}\right] - \left[c^{32}, c^{13}\right]. \tag{48}$$

In the case when $c^{12} = 0$ the corresponding r-matrix coincides with a standard rational r-matrix. That is why we will call this r-matrix to be a "shifted" rational r-matrix.

In this paper we are interested in the diagonal r-matrices $r^{12}(u, v)$ and diagonal tensors c^{12}. There is an evident simple class of diagonal non-skew-symmetric solutions of Eq. (48):

$$c^{12} = \sum_{i=1}^{n} c_i X_{ii} \otimes X_{ii}.$$

In the present paper we restrict ourselves to this class of solutions of Eq. (48).

The corresponding r-matrix (47) acquires the form:

$$r^{12}(u, v) = \sum_{i,j=1}^{n} \left(\frac{1}{u - v} + c_i \delta_{ij} \right) X_{ij} \otimes X_{ji}. \tag{49}$$

It is easy to show that for the shifted rational r-matrices (49) any constant diagonal matrix $c(u) = \sum_{i=1}^{n} c_{ii} X_{ii}$ is a shift element.

Remark 12. Observe that the "shifted" rational r-matrix was considered also in the paper [35], but the constant "shift tensor" used in this paper was a skew-symmetric one.

4.1.2. *Gaudin-type models*

For the given shifted rational r-matrices the Gaudin-type Lax operator with external magnetic field (15) has the form:

$$\hat{L}^c(u) = \sum_{m=1}^{N} \sum_{i,j=1}^{n} \left(\frac{1}{(\nu_m - u)} + c_i \delta_{ij} \right) \hat{S}_{ji}^{(m)} X_{ij} + \sum_{i=1}^{n} c_{ii} X_{ii}. \tag{50}$$

The corresponding Gaudin-type Hamiltonians in an external magnetic field (17) are written as follows:

$$\hat{H}^c_{\nu_l} = \sum_{k=1,k\neq l}^{N} \sum_{i,j=1}^{n} \left(\frac{1}{(\nu_k - \nu_l)} + \delta_{ij} c_i \right) \hat{S}^{(k)}_{ji} \hat{S}^{(l)}_{ij} + \frac{1}{2} \sum_{i,j=1}^{n} c_i \hat{S}^{(l)}_{ii} \hat{S}^{(l)}_{ii} + \sum_{i=1}^{n} c_{ii} \hat{S}^{(l)}_{ii}, \tag{51}$$

were $l \in \overline{1, N}$. The Gaudin-type Lax operators and Hamiltonians without external magnetic field are obtained by putting $c_{ii} = 0$, $i \in \overline{1, n}$ in the formulas (50), (51).

4.1.3. Bethe ansatz

Let us now consider spectrum of the Gaudin type Hamiltonians corresponding to the "shifted" rational r-matrix and the corresponding Bethe-type equations. For this purpose let us remark that shifted rational r-matrix (49) evidently satisfy the conditions (19a), (19c). That is why it is necessary only to specify the formulas (43), (44) and (45) in this case.

In the result, specifying the formula (43) we obtain:

$$h_l = h^0_l + \sum_{k=1}^{n} \lambda^{(l)}_k (c_k M_{k-1} - c_k M_k) + \sum_{k=1}^{n} \lambda^{(l)}_k \left(\sum_{j=1}^{M_{k-1}} \frac{1}{(v^{(k-1)}_j - \nu_l)} - \sum_{j=1}^{M_k} \frac{1}{(v^{(k)}_j - \nu_l)} \right), \tag{52}$$

where h^0_l is an eigen-value of these Hamiltonians on the vacuum vector obtained by specification of the formula (44):

$$h^0_l = \sum_{k=1}^{n} \sum_{m=1,m\neq l}^{N} \left(\frac{1}{(\nu_m - \nu_l)} + c_k \right) \lambda^{(m)}_k \lambda^{(l)}_k + \sum_{k=1}^{n} c_k \left(\lambda^{(l)}_k\right)^2 + \sum_{k=1}^{n} c_{kk} \lambda^{(l)}_k, \tag{53}$$

where c_{kk} are the components of the shift element — external magnetic field and we have used that $r^0_{-,k}(\nu_l, \nu_l) = r^0_{+,k}(\nu_l, \nu_l) = 0$, $r^0_{kk}(\nu_l, \nu_l) = c_k$.

The Bethe equations (45) have in this case the following form:

$$\sum_{m=1}^{N} \lambda^{(m)}_1 \left(\frac{1}{(\nu_m - v^{(1)}_i)} + c_1 \right) - \sum_{m=1}^{N} \lambda^{(m)}_2 \left(\frac{1}{(\nu_m - v^{(1)}_i)} + c_2 \right) - (c_1 + c_2) + (c_{11} - c_{22})$$
$$= \sum_{j=1; j\neq i}^{M_1} \left(\frac{2}{(v^{(1)}_j - v^{(1)}_i)} + (c_1 + c_2) \right) - \sum_{j=1}^{M_2} \left(\frac{1}{(v^{(2)}_j - v^{(1)}_i)} + c_2 \right), \tag{54a}$$

$$\sum_{m=1}^{N} \lambda^{(m)}_{k+1} \left(\frac{1}{(\nu_m - v^{(k+1)}_i)} + c_{k+1} \right) - \sum_{m=1}^{N} \lambda^{(m)}_{k+2} \left(\frac{1}{(\nu_m - v^{(k+1)}_i)} + c_{k+2} \right)$$
$$- (c_{k+1} + c_{k+2}) + (c_{k+1k+1} - c_{k+2k+2})$$
$$= \sum_{j=1; j\neq i}^{M_{k+1}} \left(\frac{2}{(v^{(k+1)}_j - v^{(k+1)}_i)} + c_{k+1} + c_{k+2} \right) - \sum_{j=1}^{M_k} \left(\frac{1}{(v^{(k)}_j - v^{(k+1)}_i)} + c_{k+1} \right)$$
$$- \sum_{j=1}^{M_{k+2}} \left(r_{k+2k+2} \frac{1}{(v^{(k+2)}_j - v^{(k+1)}_i)} + c_{k+2} \right), \quad k \in \overline{1, n-3}, \tag{54b}$$

$$\sum_{m=1}^{N}\lambda_{n-1}^{(m)}\left(\frac{1}{(\nu_m-v_i^{(n-1)})}+c_{n-1}\right)-\sum_{m=1}^{N}\lambda_{n}^{(m)}\left(\frac{1}{(\nu_m-v_i^{(n-1)})}+c_{n}\right)$$
$$-(c_{n-1}+c_n)+(c_{n-1n-1}-c_{nn})$$
$$=\sum_{j=1;j\neq i}^{M_{n-1}}\left(\frac{2}{(v_j^{(n-1)}-v_i^{(n-1)})}+c_{n-1}+c_n\right)$$
$$-\sum_{j=1}^{M_{n-2}}\left(\frac{1}{(v_j^{(n-2)}-v_i^{(n-1)})}+c_{n-1}\right). \tag{54c}$$

Finally, let us calculate the spectrum of the generating function of the linear integrals. By the virtue of the formula (46) we will have the following answer for the spectrum:

$$\lambda_n(u)=\sum_{k=1}^{n}\left(\sum_{m=1}^{N}\lambda_k^{(m)}\frac{1}{(\nu_m-u)}+c_{kk}+c_k\left(\sum_{m=1}^{N}\lambda_k^{(m)}+(M_{k-1}-M_k)\right)\right).$$

4.2. *Shifted trigonometric r-matrices and Bethe ansatz*

4.2.1. *Shifted trigonometric r-matrices*

Let us now consider skew-symmetric trigonometric $gl(n)$-valued r-matrices in the following parametrization [18]:

$$r_{trig}(u,v)=\frac{(u+v)}{(u-v)}\sum_{i=1}^{n}X_{ii}\otimes X_{ii}+\frac{2u}{u-v}\sum_{i,j=1,i<j}^{n}X_{ij}\otimes X_{ji}$$
$$+\frac{2v}{u-v}\sum_{i,j=1,j>i}^{n}X_{ij}\otimes X_{ji}. \tag{55}$$

It satisfies ordinary classical Yang–Baxter equation. It is possible to show [19] that its shifted version:

$$r(u,v)=r_{trig}(u,v)+\sum_{i,j=1}^{n}c_{ij}X_{ii}\otimes X_{jj} \tag{56}$$

satisfies the generalized classical Yang–Baxter equation for any constant tensor c_{ij}. In order to have the "diagonal" r-matrix and apply the nested algebraic Bethe ansatz we will hereafter restrict ourselves to the case $c_{ij}=c_i\delta_{ij}$.

The corresponding r-matrix acquires the following form:

$$r(u,v)=\sum_{i=1}^{n}\left(\frac{(u+v)}{(u-v)}+c_i\right)X_{ii}\otimes X_{ii}+\frac{2u}{u-v}\sum_{i,j=1,i<j}^{n}X_{ij}\otimes X_{ji}$$
$$+\frac{2v}{u-v}\sum_{i,j=1,j>i}^{n}X_{ij}\otimes X_{ji}. \tag{57}$$

Using the Cartan-invariance of the r-matrix (55) it is easy to show, that shift elements for the r-matrices (55) and (56) coincide with the arbitrary constant element of the Cartan subalgebra: $c(u)=\sum_{i=1}^{n}c_{ii}X_{ii}$.

4.2.2. Gaudin-type models

For the shifted trigonometric r-matrices the Gaudin-type Lax operator with external magnetic field (15) has the form:

$$\hat{L}^c(v) = \sum_{k=1}^{N}\sum_{i=1}^{n}\left(\frac{(\nu_k+u)}{(\nu_k-u)}+c_i\right)S_{ii}^{(k)}X_{ii} + \frac{2\nu_k}{\nu_k-u}\sum_{i,j=1,i<j}^{n} S_{ij}^{(k)}X_{ji} + \frac{2u}{\nu_k-u}\sum_{i,j=1,j>i}^{n} S_{ij}^{(k)}X_{ji} + \sum_{i=1}^{n} c_{ii}X_{ii}. \tag{58}$$

The corresponding Gaudin-type Hamiltonians in an external magnetic field (17) are written as follows:

$$\hat{H}_{\nu_l}^c = \sum_{k=1,k\neq l}^{N}\sum_{i=1}^{n}\left(\frac{(\nu_k+\nu_l)}{(\nu_k-\nu_l)}+c_i\right)S_{ii}^{(k)}S_{ii}^{(l)} + \frac{2\nu_k}{\nu_k-\nu_l}\sum_{i,j=1,i<j}^{n} S_{ij}^{(k)}S_{ji}^{(l)} + \frac{2\nu_l}{\nu_k-\nu_l}\sum_{i,j=1,j>i}^{n} S_{ij}^{(k)}S_{ji}^{(l)} + \sum_{i=1}^{n} c_i S_{ii}^{(l)}S_{ii}^{(l)} + \sum_{i=1}^{n} c_{ii}\hat{S}_{ii}^{(l)}, \tag{59}$$

were $l \in \overline{1,N}$. The Gaudin-type Lax operators and Hamiltonians without external magnetic field are obtained by putting $c_{ii}=0$, $i \in \overline{1,n}$ in the formulas (58), (59).

4.2.3. Bethe ansatz

Let us now consider spectrum of the Gaudin type Hamiltonians corresponding to the "shifted" trigonometric r-matrix and the corresponding Bethe-type equations. For this purpose let us remark that shifted trigonometric r-matrix (57) satisfy the conditions (19a), (19c). That is why it is necessary only to specify the formulas (43), (44) and (45) in this case.

The spectrum of the generalized Gaudin Hamiltonians (16) and the generalized Gaudin Hamiltonians in an external magnetic field (17) is obtained by specification of the formula (43) and has the form:

$$h_l = h_l^0 + \sum_{k=1}^{n}\left(\lambda_k^{(l)}\left(\sum_{j=1}^{M_{k-1}}\frac{(v_j^{(k-1)}+\nu_l)}{(v_j^{(k-1)}-\nu_l)} - \sum_{j=1}^{M_k}\frac{(v_j^{(k)}+\nu_l)}{(v_j^{(k)}-\nu_l)}\right) + \lambda_k^{(l)}c_k(M_{k-1}-M_k)\right), \tag{60}$$

where h_l^0 is an eigen-value of these Hamiltonians on the vacuum vector $\boldsymbol{\Omega}$:

$$h_l^0 = \sum_{k=1}^{n}\left(\sum_{m=1,m\neq l}^{N}\left(\frac{(\nu_m+\nu_l)}{(\nu_m-\nu_l)}+c_k\right)\lambda_k^{(m)}\lambda_k^{(l)} + c_k\big(\lambda_k^{(l)}\big)^2\right) + \sum_{k=1}^{n} c_{kk}\lambda_k^{(l)}, \tag{61}$$

c_{kk} are the components of the shift element — external magnetic field and the "rapidities" $v_k^{(i)}$, $k \in \overline{1,M_i}$, $i \in \overline{1,n-1}$ satisfy the following Bethe equations, which follow from the general formula (45):

$$\sum_{m=1}^{N}\lambda_1^{(m)}\left(\frac{(\nu_m+v_i^{(1)})}{(\nu_m-v_i^{(1)})}+c_1\right)-\sum_{m=1}^{N}\lambda_2^{(m)}\left(\frac{(\nu_m+v_i^{(1)})}{(\nu_m-v_i^{(1)})}+c_2\right)-(c_1+c_2)+(c_{11}-c_{22})$$
$$=\sum_{j=1;j\neq i}^{M_1}\left(\frac{2(v_j^{(1)}+v_i^{(1)})}{(v_j^{(1)}-v_i^{(1)})}+c_1+c_2\right)-\sum_{j=1}^{M_2}\left(\frac{(v_j^{(2)}+v_i^{(1)})}{(v_j^{(2)}-v_i^{(1)})}+c_2\right), \tag{62a}$$

$$\sum_{m=1}^{N}\lambda_{k+1}^{(m)}\left(\frac{(\nu_m+v_i^{(k+1)})}{(\nu_m-v_i^{(k+1)})}+c_{k+1}\right)-\sum_{m=1}^{N}\lambda_{k+2}^{(m)}\left(\frac{(\nu_m+v_i^{(k+1)})}{(\nu_m-v_i^{(k+1)})}+c_{k+2}\right)$$
$$-(c_{k+1}+c_{k+2})+(c_{k+1k+1}-c_{k+2k+2})$$
$$=\sum_{j=1;j\neq i}^{M_{k+1}}\left(\frac{2(v_j^{(k+1)}+v_i^{(k+1)})}{(v_j^{(k+1)}-v_i^{(k+1)})}+c_{k+1}+c_{k+2}\right)-\sum_{j=1}^{M_k}\left(\frac{(v_j^{(k)}+v_i^{(k+1)})}{(v_j^{(k)}-v_i^{(k+1)})}+c_{k+1}\right)$$
$$-\sum_{j=1}^{M_{k+2}}\left(\frac{(v_j^{(k+2)}+v_i^{(k+1)})}{(v_j^{(k+2)}-v_i^{(k+1)})}+c_{k+2}\right),\quad k\in\overline{1,n-3}, \tag{62b}$$

$$\sum_{m=1}^{N}\lambda_{n-1}^{(m)}\left(\frac{(\nu_m+v_i^{(n-1)})}{(\nu_m-v_i^{(n-1)})}+c_{n-1}\right)-\sum_{m=1}^{N}\lambda_{n}^{(m)}\left(\frac{(\nu_m+v_i^{(n-1)})}{(\nu_m-v_i^{(n-1)})}+c_{n}\right)$$
$$-(c_{n-1}+c_n)+(c_{n-1n-1}-c_{nn})$$
$$=\sum_{j=1;j\neq i}^{M_{n-1}}\left(\frac{2(v_j^{(n-1)}+v_i^{(n-1)})}{(v_j^{(n-1)}-v_i^{(n-1)})}+c_{n-1}+c_n\right)$$
$$-\sum_{j=1}^{M_{n-2}}\left(\frac{(v_j^{(n-2)}+v_i^{(n-1)})}{(v_j^{(n-2)}-v_i^{(n-1)})}+c_{n-1}\right), \tag{62c}$$

where we have used that in this case $r^0_{-,k}(\nu_l,\nu_l)+r^0_{+,k}(\nu_l,\nu_l)=0$ and $r^0_{kk}(\nu_l,\nu_l)=c_k$.

Finally let us calculate the spectrum of the generating function of the linear integrals. By the virtue of the formula (46) we will have the following answer for the spectrum:

$$\lambda_n(u)=\sum_{k=1}^{n}\left(\sum_{m=1}^{N}\lambda_k^{(m)}\frac{(\nu_m-u)}{(\nu_m+u)}+c_{kk}+c_k\left(\sum_{m=1}^{N}\lambda_k^{(m)}+(M_{k-1}-M_k)\right)\right).$$

4.3. Shifted Z_2-graded r-matrices and Bethe ansatz

4.3.1. Shifted Z_2-graded r-matrices

Let σ be an automorphism of the second order of the Lie algebra $gl(n)$. Let $gl(n)=gl(n)_{\bar 0}+gl(n)_{\bar 1}$ be the corresponding Z_2-grading of $\mathfrak{g}$, i.e.:

$$[gl(n)_{\bar 0},gl(n)_{\bar 0}]\subset gl(n)_{\bar 0},\qquad [gl(n)_{\bar 0},gl(n)_{\bar 1}]\subset gl(n)_{\bar 1},\qquad [gl(n)_{\bar 1},gl(n)_{\bar 1}]\subset gl(n)_{\bar 0}$$

Let $\Omega^{12}_{\bar 0}$ be the tensor Casimir on $gl(n)_{\bar 0}$, $c^{12}_{\bar 0}$ be a constant non-skew-symmetric solution of the "modified" Yang–Baxter equation on $gl(n)_{\bar 0}$:

$$[c^{12}_{\bar 0},c^{13}_{\bar 0}]-[c^{23}_{\bar 0},c^{12}_{\bar 0}]+[c^{32}_{\bar 0},c^{13}_{\bar 0}]=[\Omega^{32}_{\bar 0},c^{13}_{\bar 0}]. \tag{63}$$

Then it is possible to show [19] that the function

$$r(u,v)=\frac{v^2}{u^2-v^2}\Omega^{12}_{\bar{0}}+\frac{uv}{u^2-v^2}\Omega^{12}_{\bar{1}}+c^{12}_{\bar{0}}, \tag{64}$$

where $\Omega^{12}_{\bar{0}}=\sum_{\alpha=1}^{\dim gl(n)_{\bar{0}}}X_{\bar{0},\alpha}\otimes X^{\bar{0},\alpha}$, $\Omega^{12}_{\bar{1}}=\sum_{\alpha=1}^{\dim gl(n)_{\bar{1}}}X_{\bar{1},\alpha}\otimes X^{\bar{1},\alpha}$, $X_{\bar{j},\alpha}$ and $X^{\bar{j},\alpha}$ are dual bases in $gl(n)_{\bar{j}}$, satisfies the generalized classical Yang–Baxter equation.

In order to satisfy the conditions imposed on the r-matrix by the nested Bethe ansatz we will further specify Z_2-grading of $gl(n)$. In more details, we will assume that

$$gl(n)_{\bar{0}}=\mathrm{Span}_{\mathbb{C}}\{X_{11},X_{ij}\mid i,j\in\overline{2,n}\}=gl(1)\oplus gl(n-1),$$
$$gl(n)_{\bar{1}}=\mathrm{Span}_{\mathbb{C}}\{X_{1i},X_{j1}\mid i,j\in\overline{2,n}\}.$$

Due to the fact that the element X_{11} belongs to center of $gl(n)_{\bar{0}}$, there exists an evident family of solutions of Eq. (63) on $gl(n)_{\bar{0}}$, namely the tensor $c^{12}_{\bar{0}}=c_1X_{11}\otimes X_{11}$. For the further convenience we will divide the r-matrix by v^2, which is the symmetry transformation for non-skew-symmetric classical r-matrices.

The corresponding r-matrix (64) (after the division on v^2) acquires the following form:

$$r(u,v)=\left(\frac{1}{u^2-v^2}+\frac{c_1}{v^2}\right)X_{11}\otimes X_{11}+\frac{1}{u^2-v^2}\sum_{i,j=2}^{n}X_{ij}\otimes X_{ji}$$
$$+\frac{u}{v(u^2-v^2)}\sum_{i=2}^{n}(X_{i1}\otimes X_{1i}+X_{1i}\otimes X_{i1}). \tag{65}$$

Remark 13. Observe, that in the partial case $c_1=0$ the r-matrix (65) is connected with the reflection equation algebra (see [20]).

It is possible to show that shift element $c(u)$ for the r-matrices (65) should belong to the center of $gl(n)_{\bar{0}}$ and has the form: $c(u)=\frac{1}{u^2}(c_{11}X_{11}+c_{22}\sum_{j=2}^{n}X_{jj})$.

4.3.2. Gaudin-type models

For the given shifted Z_2-graded r-matrices (65) the Gaudin-type Lax operator with external magnetic field (15) has the form:

$$\hat{L}^c(v)=\sum_{k=1}^{N}\left(\left(\frac{1}{\nu_k^2-v^2}+\frac{c_1}{v^2}\right)S^{(k)}_{11}X_{11}+\frac{1}{\nu_k^2-v^2}\sum_{i,j=2}^{n}S^{(k)}_{ij}X_{ji}\right.$$
$$\left.+\frac{\nu_k}{v(\nu_k^2-v^2)}\sum_{i=2}^{n}\left(S^{(k)}_{i1}X_{1i}+S^{(k)}_{1i}X_{i1}\right)\right)$$
$$+\frac{1}{v^2}\left(c_{11}X_{11}+c_{22}\sum_{j=2}^{n}X_{jj}\right). \tag{66}$$

The corresponding Gaudin-type Hamiltonians in an external magnetic field (17) are written as follows:

$$\hat{H}^c_{\nu_l} = \sum_{k=1,k\neq l}^{N}\Bigg(\bigg(\frac{1}{\nu_k^2-\nu_l^2}+\frac{c_1}{\nu_l^2}\bigg)S^{(k)}_{11}S^{(l)}_{11}+\frac{1}{\nu_k^2-\nu_l^2}\sum_{i,j=2}^{n}S^{(k)}_{ij}S^{(l)}_{ji}$$
$$+\frac{\nu_k}{\nu_l(\nu_k^2-\nu_l^2)}\sum_{i=2}^{n}\big(S^{(k)}_{i1}S^{(l)}_{1i}+S^{(k)}_{1i}S^{(l)}_{i1}\big)\Bigg)+\frac{c_1}{\nu_l^2}S^{(l)}_{11}S^{(l)}_{11}$$
$$+\frac{1}{2\nu_l^2}\sum_{i=2}^{n}\big(S^{(l)}_{i1}S^{(l)}_{1i}+S^{(l)}_{1i}S^{(l)}_{i1}\big)+\frac{1}{\nu_l^2}\Bigg(c_{11}S^{(l)}_{11}+c_{22}\sum_{j=2}^{n}S^{(l)}_{jj}\Bigg), \tag{67}$$

where $l\in\overline{1,N}$ and we have used that the regular part of the r-matrix (65) (with respect to the new spectral parameters u^2 and v^2) has the following form:

$$r^0(u,u)=\frac{c_1}{u^2}X_{11}\otimes X_{11}+\frac{1}{2u^2}\sum_{i=2}^{n}(X_{i1}\otimes X_{1i}+X_{1i}\otimes X_{i1}). \tag{68}$$

The Gaudin-type Lax operators and Hamiltonians without external magnetic field are obtained by putting $c_{ii}=0$, $i\in\overline{1,2}$ in the formulas (66), (67).

4.3.3. Bethe ansatz

Let us now consider spectrum of the Gaudin type Hamiltonians corresponding to the "shifted" Z_2-graded r-matrix and the corresponding Bethe-type equations. For this purpose let us remark that shifted Z_2-graded r-matrix (65) satisfy the conditions (19a), (19b). That is why it is necessary only to specify the formulas (43), (44) and (45) in the case.

The spectrum of the Hamiltonians (17) is obtained by specification of the formula (43):

$$h_l=h_l^0-\sum_{j=1}^{M_1}\lambda_1^{(l)}\bigg(\frac{1}{(v_j^{(1)})^2-\nu_l^2}+\frac{c_1}{\nu_l^2}\bigg)$$
$$+\sum_{k=2}^{n}\lambda_k^{(l)}\Bigg(\sum_{j=1}^{M_{k-1}}\frac{1}{(v_j^{(k-1)})^2-\nu_l^2}-\sum_{j=1}^{M_k}\frac{1}{(v_j^{(k)})^2-\nu_l^2}\Bigg), \tag{69}$$

where h_l^0 is an eigen-value of these Hamiltonians on the vacuum vector $\boldsymbol{\Omega}$:

$$h_l^0=\sum_{m=1,m\neq l}^{N}\bigg(\frac{1}{\nu_m^2-\nu_l^2}+\frac{c_1}{\nu_l^2}\bigg)\lambda_1^{(m)}\lambda_1^{(l)}+\frac{c_1}{\nu_l^2}\big(\lambda_1^{(l)}\big)^2+\sum_{k=2}^{n}\sum_{m=1,m\neq l}^{N}\frac{\lambda_k^{(m)}\lambda_k^{(l)}}{(\nu_m^2-\nu_l^2)}$$
$$+\frac{1}{2\nu_l^2}\Bigg((n-1)\lambda_1^{(l)}-\sum_{s=2}^{n}\lambda_s^{(l)}\Bigg)+\frac{c_{11}}{\nu_l^2}\lambda_1^{(l)}+\frac{c_{22}}{\nu_l^2}\sum_{k=2}^{n}\lambda_k^{(l)}, \tag{70}$$

$\frac{c_{kk}}{\nu_l^2}$ are the components of the shift element — external magnetic field and we have used that, as it follows from the explicit form of the regular part of the r-matrix (68):

$$r^0_{-,1}(u,u)=r^0_{+,1}(u,u)=\frac{1}{2u^2},\qquad r^0_{11}(u,u)=\frac{c_1}{u^2},$$
$$r^0_{-,k}(u,u)=r^0_{+,k}(u,u)=0,\qquad r^0_{kk}(u,u)=0,\quad \forall k>1.$$

The "rapidities" $v_k^{(i)}$, $k\in\overline{1,M_i}$, $i\in\overline{1,n-1}$ satisfy the Bethe-type equations the form of which follows from the general equations (45):

$$\sum_{m=1}^{N}\lambda_1^{(m)}\left(\frac{1}{\nu_m^2-(v_i^{(1)})^2}+\frac{c_1}{(v_i^{(1)})^2}\right)-\sum_{m=1}^{N}\frac{\lambda_2^{(m)}}{\nu_m^2-(v_i^{(1)})^2}$$
$$+\left(\frac{c_{11}}{(v_i^{(1)})^2}-\frac{c_{22}}{(v_i^{(1)})^2}\right)-\frac{c_1}{(v_i^{(1)})^2}+\frac{n}{2(v_i^{(1)})^2}$$
$$=\sum_{j=1;\,j\neq i}^{M_1}\left(\frac{2}{(v_j^{(1)})^2-(v_i^{(1)})^2}+\frac{c_1}{(v_i^{(1)})^2}\right)-\sum_{j=1}^{M_2}\frac{1}{(v_j^{(2)})^2-(v_i^{(1)})^2}, \tag{71a}$$

$$\sum_{m=1}^{N}\frac{\lambda_{k+1}^{(m)}}{\nu_m^2-(v_i^{(k+1)})^2}-\sum_{m=1}^{N}\frac{\lambda_{k+2}^{(m)}}{\nu_m^2-(v_i^{(k+1)})^2}$$
$$=\sum_{j=1;\,j\neq i}^{M_{k+1}}\frac{2}{(v_j^{(k+1)})^2-(v_i^{(k+1)})^2}-\sum_{j=1}^{M_k}\frac{1}{(v_j^{(k)})^2-(v_i^{(k+1)})^2}$$
$$-\sum_{j=1}^{M_{k+2}}\frac{1}{(v_j^{(k+2)})^2-(v_i^{(k+1)})^2},\quad k\in\overline{1,n-3}, \tag{71b}$$

$$\sum_{m=1}^{N}\frac{\lambda_{n-1}^{(m)}}{\nu_m^2-(v_i^{(n-1)})^2}-\sum_{m=1}^{N}\frac{\lambda_{n}^{(m)}}{\nu_m^2-(v_i^{(n-1)})^2}$$
$$=\sum_{j=1;\,j\neq i}^{M_{n-1}}\frac{2}{(v_j^{(n-1)})^2-(v_i^{(n-1)})^2}-\sum_{j=1}^{M_{n-2}}\frac{1}{(v_j^{(n-2)})^2-(v_i^{(n-1)})^2}. \tag{71c}$$

Remark 14. Observe that among Eqs. (71) only Eqs. (71a) are different from those for the standard "unshifted" rational r-matrix. Observe also that for the case $c_1=0$ Eqs. (71) coincide with those obtained in our previous paper [20] by analytical Bethe ansatz method.

Due to the fact that the considered r-matrix satisfy the conditions (19b) the spectrum of $\hat{t}_n(u)$ has the following simple form:

$$\lambda_n(u)=\sum_{k=1}^{n}\sum_{m=1}^{N}\frac{\lambda_k^{(m)}}{\nu_m^2-u^2}+\frac{c_{11}}{u^2}+(n-1)\frac{c_{22}}{u^2}+c_1\left(\sum_{m=1}^{N}\lambda_1^{(m)}-M_1\right).$$

5. Conclusion and discussion

In the present paper we have considered quantum integrable systems associated with the Lie algebra $gl(n)$ and Cartan-invariant non-skew-symmetric classical r-matrices. We have described the sub-class of Cartan-invariant non-skew-symmetric classical r-matrices for which exists the standard procedure of nested Bethe ansatz and diagonalized the corresponding quantum integrable systems by its means. We illustrate the obtained results by the examples of the generalized Gaudin systems associated with three classes of non-skew-symmetric classical r-matrices.

It would be very interesting to classify all Cartan-invariant classical r-matrices which satisfy the conditions (19a), (19b) (or (19a), (19c)) for which the standard nested Bethe ansatz is applicable.

Another interesting and physically meaningful problem is a construction of the other (than generalized Gaudin) integrable systems associated with the considered in this paper classical r-matrices. Some results in this direction are already obtained and soon be published.

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